Theoretical Computer Science 520 (2014) 97–110

Contents lists available at ScienceDirect

Theoretical Computer Science

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Strong matching preclusion for torus networks ☆

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ARTICLE INFO

Article history:
Received 6 September 2012
Received in revised form 26 April 2013
Accepted 24 October 2013
Communicated by S.-y. Hsieh

Keywords:
Interconnection networks
Torus
Cartesian product
Strong matching preclusion

ABSTRACT

The torus network is one of the most popular interconnection network topologies for massively parallel computing systems. Strong matching preclusion that additionally permits more destructive vertex faults in a graph is a more extensive form of the original matching preclusion that assumes only edge faults. In this paper, we establish the strong matching preclusion number and all minimum strong matching preclusion sets for bipartite torus networks and 2-dimensional nonbipartite torus networks.

1. Introduction

A matching of a graph is a set of pairwise nonadjacent edges. For a graph with n vertices, a matching M is called perfect if its size $|M| = \frac{n}{2}$ for even n, or almost perfect if $|M| = \frac{n-1}{2}$ for odd n. A graph is matchable if it has either a perfect matching or an almost perfect matching. Otherwise, it is called unmatchable. Throughout the paper, we only consider simple and even graphs, that is, graphs with an even number of vertices with no parallel edges or loops. For graph-theoretical terminology and notation not defined here we follow [4]. Let $G = (V(G), E(G))$ be a graph. A set F of edges in G is called a matching preclusion set (MP set for short) if $G - F$ has neither a perfect matching nor an almost perfect matching. The matching preclusion number of G (MP number for short), denoted by $mp(G)$, is defined to be the minimum size of all possible such sets of G. The minimum MP set of G is any MP set whose size is $mp(G)$. A matching preclusion set of a graph is trivial if all its edges are incident to a single vertex.

Since the problem of matching preclusion was first presented by Brigham et al. [3], several classes of graphs have been studied to understand their matching preclusion properties [5–8,11,13,14]. An obvious application of the matching preclusion problem was addressed in [3]: when each node of interconnection networks is demanded to have a special partner at any time, those that have larger matching preclusion numbers will be more robust in the event of link failures.

Another form of matching obstruction, which is in fact more offensive, is through node failures. As an extensive form of matching preclusion, the problem of strong matching preclusion was proposed by Park and Ihm in [12]. A set F of vertices and/or edges in a matchable graph G is called a strong matching preclusion set (SMP set for short) if $G - F$ has neither a perfect matching nor an almost perfect matching. The strong matching preclusion number (SMP number for short) of G, denoted by $smp(G)$, is defined to be the minimum size of all possible such sets of G. The minimum SMP set of G is any

☆ This work is supported by the National Natural Science Foundation of China (61370001) and the Doctoral Fund of Ministry of Education of China (20111401110005).

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0304-3975/$ – see front matter
http://dx.doi.org/10.1016/j.tcs.2013.10.023

SMP set whose size is $smp(G)$. Note that the strong matching preclusion is more general than the problems discussed in [1,9], which considered only vertex deletions.

Specially, when G itself does not contain perfect matchings or almost perfect matchings, both $smp(G)$ and $mp(G)$ are regarded as zero. These numbers are undefined for a trivial graph with only one vertex. Notice that an MP set of a graph is a special SMP set of the graph.

Proposition 1.1. *(See [12].) For every nontrivial graph G, $smp(G) \leqslant mp(G)$.*

However, the strong matching preclusion numbers did not decrease for such graphs as restricted hypercube-like graphs and recursive circulants [12]. Then, followed by this work, the strong matching preclusion problem was studied for some classes of graphs such as alternating group graphs and split-stars [2].

When a set F of vertices and/or edges is removed from a graph, the set is called a fault set. Let F_v and F_e be the fault vertex set and the fault edge set, respectively. We have $F = F_v \cup F_e$. For any vertex $v \in V(G)$, let $N_G(v)$ be all neighbouring vertices adjacent to v and let $I_G(v)$ be all edges incident to v. Clearly, a fault set, which separates exactly one isolated vertex from the remaining even graph, forms a simple SMP set of the original graph.

Proposition 1.2. *(See [12].) Let G be a graph. Given a fault vertex set $X(v) \subseteq N_G(v)$ and a fault edge set $Y(v) \subseteq I_G(v)$, $X(v) \cup Y(v)$ is an SMP set of G if* (i) *$w \in X(v)$ if and only if $(v, w) \notin Y(v)$ for every $w \in N_G(v)$, and* (ii) *the number of vertices in $G - (X(v) \cup Y(v))$ is even.*

The above proposition suggests an easy way of building SMP sets. Any SMP set constructed as specified in Proposition 1.2 is called trivial. If $smp(G) = \delta(G)$, then G is called maximally strong matched. If every minimum SMP set of G is trivial, then G is called super strong matched. It is easy to see that, for an arbitrary vertex of degree at least one, there always exists a trivial SMP set which isolates the vertex. This observation leads to the following fact.

Proposition 1.3. *(See [12].) For any graph G with no isolated vertices, $smp(G) \leqslant \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

2. Definitions and terminology

The torus forms a basic class of interconnection networks. Let G and H be two simple graphs. Their Cartesian product $G \times H$ is the graph with vertex set $V(G) \times V(H) = \{gh\colon\ g \in V(G),\ h \in V(H)\}$, in which two vertices g_1h_1 and g_2h_2 are adjacent if and only if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $(g_1, g_2) \in E(G)$ and $h_1 = h_2$. For $n \geqslant 3$, let $G_1, G_2, \ldots, G_n$ be n simple graphs. Similarly, the Cartesian product $G_1 \times G_2 \times \cdots \times G_n$ can be defined. It is easy to see that "×" is associative and commutative under isomorphism. Let C_k be the cycle of length k with the vertex set $\{0, 1, \ldots, k-1\}$. Two vertices $u, v \in V(C_k)$ are adjacent in C_k if and only if $u = v \pm 1 \pmod k$. The torus $T(k_1, k_2, \ldots, k_n)$ with $n \geqslant 2$ and $k_i \geqslant 3$ for all i is defined to be $T(k_1, k_2, \ldots, k_n) = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}$ with the vertex set $\{u_1u_2 \ldots u_n\colon\ u_i \in \{0, 1, \ldots, k_i - 1\},\ 1 \leqslant i \leqslant n\}$. Two vertices $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ are adjacent in $T(k_1, k_2, \ldots, k_n)$ if and only if there exists some $j \in \{1, 2, \ldots, n\}$ such that $u_j = v_j \pm 1 \pmod{k_j}$ and $u_i = v_i$ for $i \in \{1, 2, \ldots, n\} \backslash \{j\}$. Clearly, $T(k_1, k_2, \ldots, k_n)$ is a connected $2n$-regular graph consisting of $k_1k_2 \ldots k_n$ vertices. Note that we only consider even graphs in this paper, which implies that at least one of $k_1, k_2, \ldots, k_n$ is even.

Let $T(k_1, k_2)$ be a 2-dimensional torus, where $k_1 \geqslant 3$ and $k_2 \geqslant 3$. Then $T(k_1, k_2) = C_{k_1} \times C_{k_2}$. We view $C_{k_1} \times C_{k_2}$ as consisting of k_2 copies of C_{k_1}. Let these copies be $C_{k_1}^0, C_{k_1}^1, \ldots, C_{k_1}^{k_2-1}$ labeled along the cycle C_{k_2}. The edges between different copies of C_{k_1} are called cross edges. Denote the set of cross edges between $C_{k_1}^i$ and $C_{k_1}^{i+1 \pmod{k_2}}$ by $M_{i,i+1 \pmod{k_2}}$ for $0 \leqslant i \leqslant k_2 - 1$. For clarity of presentation, we omit writing "(mod k_2)" in similar expressions for the remainder of the paper. Clearly, each of these sets is a matching saturating all vertices of the corresponding copies of C_{k_1}. For convenience, a vertex with subscript 0 (e.g. x_0) will denote a vertex in $C_{k_1}^0$, the corresponding vertex with subscript 1 (e.g. x_1) will denote the vertex in $C_{k_1}^1$ which is adjacent to this vertex via a cross edge, etc., and the corresponding vertex with subscript $k_2 - 1$ (e.g. x_{k_2-1}) will denote the vertex in $C_{k_1}^{k_2-1}$ which is adjacent to this vertex via a cross edge. The vertices $x_0, x_1, \ldots, x_{k_2-1}$ and the cross edges between them form a cycle of length k_2, which is denoted by $C_{k_2}(x_i)$ for some $i \in \{0, 1, \ldots, k_2 - 1\}$. For any matching M_i in $C_{k_1}^i$, the matching M_j, which satisfies that $(x_j, y_j) \in M_j$ if and only if $(x_i, y_i) \in M_i$, is called the corresponding matching to M_i.

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. A path is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. The length of a path is the number of its edges. The path is odd or even according to the parity of its length. For notational simplicity, denote by $|G|$ the number of vertices in a graph G. Let G_1 and G_2 be two graphs. $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

In this paper, we investigate the problem of strong matching preclusion for torus networks. We establish the strong matching preclusion number and all possible minimum strong matching preclusion sets for bipartite torus networks and 2-dimensional nonbipartite torus networks.

3. Main results

Lemma 3.1. *(See [12].) For a connected m-regular bipartite graph G with $m \geqslant 3$, $smp(G) = 2$. Furthermore, each of its minimum SMP sets is a set of two vertices from the same partite set.*

Theorem 3.1. *Let $k_1, k_2, \dots, k_n$ be even integers with $k_i \geqslant 4$ for each $i = 1, 2, \dots, n$. Then $T(k_1, k_2, \dots, k_n)$ is bipartite and $smp(T(k_1, k_2, \dots, k_n)) = 2$. Furthermore, each of its minimum SMP sets is a set of two vertices from the same partite set.*

Proof. Let $V_1 = \{u_1u_2 \dots u_n\colon u_1u_2 \dots u_n \in V(T(k_1, k_2, \dots, k_n))$ and $\sum_{i=1}^{n} u_i = 0 \pmod 2\}$ and $V_2 = V(T(k_1, k_2, \dots, k_n)) \setminus V_1$. Without loss of generality, let $u_1u_2 \dots u_n \in V_1$ and $v_1v_2 \dots v_n \in N_{T(k_1,k_2,\dots,k_n)}(u_1u_2 \dots u_n)$. By the definition of $T(k_1, k_2, \dots, k_n)$, there exists some $j \in \{1, 2, \dots, n\}$ such that $u_j = v_j \pm 1 \pmod{k_j}$ and $u_i = v_i$ for $i \in \{1, 2, \dots, n\} \setminus \{j\}$. If $k_1, k_2, \dots, k_n$ are even, then u_j and v_j have different parities, which implies that $\sum_{i=1}^{n} u_i$ and $\sum_{i=1}^{n} v_i$ have different parities. So $v_1v_2 \dots v_n \in V_2$, which implies that two arbitrary vertices in V_1 are nonadjacent. Similarly, two arbitrary vertices in V_2 are nonadjacent. Thus, $T(k_1, k_2, \dots, k_n)$ is bipartite. Note that $T(k_1, k_2, \dots, k_n)$ is a connected $2n$-regular graph with $2n > 3$. By Lemma 3.1, $smp(T(k_1, k_2, \dots, k_n)) = 2$ and each of its minimum SMP sets is a set of two vertices from the same partite set. □

Theorem 3.2. *Let $k \geqslant 3$ be an integer and let C_k be a cycle of length k. Then $smp(C_k) = 2$.*

Proof. By Proposition 1.3, $smp(C_k) \leqslant \delta(C_k) = 2$. Next, consider a fault set F with $|F| = 1$. If F consists of one edge, $C_k - F$ is a path of length $k - 1$. If F consists of one vertex, $C_k - F$ is a path of length $k - 2$. Note that an odd path has a perfect matching, while an even path has an almost perfect matching. We have that $C_k - F$ is matchable, which means $smp(C_k) > 1$. Therefore, $smp(C_k) = 2$. □

By Theorem 3.2, C_k is maximally strong matched, where $k \geqslant 3$. However, C_k is not super strong matched. For example, let $C = (0, 1, 2, 3, 4, 5, 0)$ be a cycle and let $F = \{(0, 5), (2, 3)\}$. It is easy to see that there is no perfect matching in $C - F$ and F is not a trivial strong matching preclusion set.

Lemma 3.2. *(See [6].) Let $T(k_1, k_2, \dots, k_n)$ be a torus with an even number of vertices. Then $mp(T(k_1, k_2, \dots, k_n)) = 2n$ and each of its minimum MP sets is trivial.*

Theorem 3.3. *Let $k_1 \geqslant 6$ be an even integer and let $k_2 \geqslant 3$ be an odd integer. Then $smp(T(k_1, k_2)) = 4$. Moreover, $T(k_1, k_2)$ is super strong matched.*

Proof. $T(k_1, k_2) = C_{k_1} \times C_{k_2}$ is a connected 4-regular graph consisting of k_1k_2 vertices. Let $F = F_v \cup F_e$ be a fault set in $T(k_1, k_2)$ such that $|F| \leqslant 4$, where F_v and F_e are the fault vertex set and the fault edge set, respectively. To prove our main result, it is enough to show that either $T(k_1, k_2) - F$ is matchable or F is a trivial strong matching preclusion set.

We define an approach to find a perfect matching in $T(k_1, k_2) - F$ as follows: we find a fault-free matching saturating some copies of C_{k_1}, in which cross edges may be used. If each remaining copy has a fault-free perfect matching, then we can extend this matching to a perfect matching in $T(k_1, k_2) - F$ by adding a fault-free matching saturating the remaining copies of C_{k_1}. This method will be called completing the matching.

We consider five cases depending on the value of $|F_v|$. Without loss of generality, assume that $|F_v \cap V(C_{k_1}^0)| \geqslant |F_v \cap V(C_{k_1}^i)|$ for $i = 1, 2, \dots, k_2 - 1$.

Case 1. $|F_v| = 4$, which means $|F_e| = 0$.

Case 1.1. $|F_v \cap V(C_{k_1}^0)| = 4$.

Since $k_1 \geqslant 6$ is an even integer, $C_{k_1}^i$ has a perfect matching for $i = 0, 1, \dots, k_2 - 1$. If $C_{k_1}^0 - F_v$ can be partitioned into a set of paths of length one, then there exists a matching M_0 saturating $C_{k_1}^0 - F_v$ and completing M_0 gives a perfect matching of $T(k_1, k_2) - F$.

Assume that $C_{k_1}^0 - F_v$ can be partitioned into a set of paths of length one plus some single vertices. Denote by s the number of these single vertices. It is easy to see that $0 < s \leqslant 4$. Since $|V(C_{k_1}^0) \setminus F_v|$ is even, s is even. Let $F_v = \{x_0, y_0, u_0, w_0\}$. We consider two subcases.

Case 1.1.1. $s = 2$.

Assume that $C_{k_1}^0 - F_v$ can be partitioned into the set M_0^* of paths of length one plus two single vertices, each of which is adjacent to one of the fault vertices, say x_0 (see Fig. 1(a)) (when $k_1 = 6$, $M_0^* = \emptyset$). Let a_0, b_0 be the two single

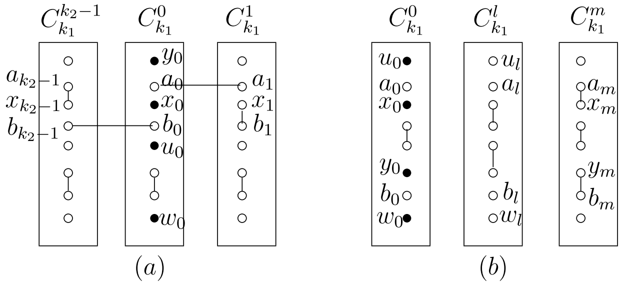

Fig. 1. $|F_v \cap V(C_{k_1}^0)| = 4$ and $s = 2$.

vertices. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{y_0, u_0, w_0\}$. Let M_i^* be the corresponding matching to M_0^* for $i = 1, 2, \ldots, k_2 - 1$. Let M_{t_0} be the matching saturating $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{a_0, b_0, x_0\}$ (when $k_2 = 3$, $M_{t_0} = \emptyset$). Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup (\bigcup_{z \in \{y_0, u_0, w_0\}} M_z) \cup (\bigcup_{t_0 \in \{a_0, b_0, x_0\}} M_{t_0}) \cup \{(a_{k_2-1}, x_{k_2-1}), (a_0, a_1), (b_{k_2-1}, b_0), (x_1, b_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that $C_{k_1}^0 - F_v$ can be partitioned into the set of paths of length one plus two single vertices, both of which have no common neighbours in F_v (see Fig. 1(b)). Let a_0, b_0 be the two single vertices. Let P_0 be a path in $C_{k_1}^0$ from a_0 to b_0 such that $x_0, y_0 \in V(P_0)$ and $u_0, w_0 \notin V(P_0)$, where x_0 and y_0 are the neighbours of a_0 and b_0, respectively. Then P_0 is an odd path. There exists $l \in \{1, \ldots, k_2 - 1\}$ such that both x_l and y_l are not fault vertices. $C_{k_2}(x_0) - \{x_0, x_l\}$ can be partitioned into the set M_{x_0} of paths of length one plus one single vertex x_m. $C_{k_2}(y_0) - \{y_0, y_l\}$ can be partitioned into the set M_{y_0} of paths of length one plus one single vertex y_m. $C_{k_1}^0 - (F_v \cup \{a_0, b_0\})$ can be partitioned into the set M_0^* of paths of length one. Let M_i^* be the corresponding matching to M_0^* for each $i \in \{1, 2, \ldots, k_2 - 1\} \backslash \{l\}$. Let M_l^* be a perfect matching in $C_{k_1}^l - \{u_l, w_l, a_l, b_l\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{u_0, w_0, a_m, b_m\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup (\bigcup_{z \in \{x_0, y_0, u_0, w_0, a_m, b_m\}} M_z) \cup \{(a_m, x_m), (b_m, y_m)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.1.2. $s = 4$.

$C_{k_1}^0 - F_v$ can be partitioned into the set M_0^* of paths of length one plus four single vertices such that two of the single vertices are adjacent to one of the fault vertices (say x_0) and the other two single vertices are adjacent to another fault vertex (say y_0) (when $k_1 = 8$, $M_0^* = \emptyset$). Let $N_{C_{k_1}^0}(x_0) = \{a_0, b_0\}$ and $N_{C_{k_1}^0}(y_0) = \{c_0, d_0\}$. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{u_0, w_0\}$. Let M_i^* be the corresponding matching to M_0^* for $i = 1, 2, \ldots, k_2 - 1$. Let M_{t_0} be the matching saturating $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{a_0, b_0, c_0, d_0, x_0, y_0\}$ (when $k_2 = 3$, $M_{t_0} = \emptyset$). Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup (\bigcup_{z \in \{u_0, w_0\}} M_z) \cup (\bigcup_{t_0 \in \{a_0, b_0, c_0, d_0, x_0, y_0\}} M_{t_0}) \cup \{(a_{k_2-1}, x_{k_2-1}), (a_0, a_1), (b_{k_2-1}, b_0), (x_1, b_1)\} \cup \{(c_{k_2-1}, y_{k_2-1}), (c_0, c_1), (d_{k_2-1}, d_0), (y_1, d_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.2. $|F_v \cap V(C_{k_1}^0)| = 3$.

There is one faulty vertex in $T(k_1, k_2) - V(C_{k_1}^0)$. Without loss of generality, assume that $F_v \cap V(C_{k_1}^i) = \{w_i\}$. Let $F_v \cap V(C_{k_1}^0) = \{x_0, y_0, z_0\}$. Assume that $C_{k_1}^0 - F_v$ can be partitioned into a set of paths of length one plus some single vertices. Denote by s the number of these single vertices. It is easy to see that $1 \leqslant s \leqslant 3$. Since $|V(C_{k_1}^0) \backslash F_v|$ is odd, $s \neq 2$. We consider two subcases.

Case 1.2.1. $s = 1$.

There exists exactly one even path P_0 in $C_{k_1}^0 - F$. If w_0 is a terminal vertex of P_0, then there exists a matching M_0 saturating $C_{k_1}^0 - \{x_0, y_0, z_0, w_0\}$. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{x_0, y_0, z_0, w_i\}$. Let M_i be the corresponding matching to M_0 for $i = 1, 2, \ldots, k_2 - 1$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{x_0, y_0, z_0, w_i\}} M_z)$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that $|P_0| = 3$ and w_0 is the internal vertex of P_0. Then there exists a matching M_0 saturating $C_{k_1}^0 - (V(P_0) \cup F)$ (when $k_1 = 6$, $M_0 = \emptyset$). Let v_0 be a terminal vertex of P_0. If $i - 1$ is even, then let $P^* = P_0 \cup \{v_i\} \cup (\bigcup_{j=1}^{i-1} (C_{k_1}^j - (v_j, w_j))) \cup \{(v_0, v_1), (w_1, w_2), \ldots, (v_{i-1}, v_i)\}$. If $i - 1$ is odd, then let $P^* = P_0 \cup \{v_i\} \cup (\bigcup_{j=i+1}^{k_2-1} (C_{k_1}^j - (v_j, w_j))) \cup \{(v_0, v_{k_2-1}), (w_{k_2-1}, w_{k_2-2}), \ldots, (v_{i+1}, v_i)\}$. Note that P^* is a fault-free odd path. So there exists a perfect matching M^* in P^*. Let M_i be a perfect matching in $C_{k_1}^i - \{w_i, v_i\}$. Then $M_0 \cup M^* \cup M_i$ or completing $M_0 \cup M^* \cup M_i$ gives a perfect matching of $T(k_1, k_2) - F$.

Assume that $|P_0| \geqslant 5$. Then there exist a terminal vertex v_0 of P_0 and $u_0 \in V(P_0)$ such that $(u_0, v_0) \in E(P_0)$ and $w_0 \neq u_0$. Note that there exists a vertex $a \in \{u_i, v_i\}$ such that $C_{k_1}^i - \{w_i, a\}$ can be partitioned into a set of paths of length one. If $a = v_i$ and $i - 1$ is even, then let $P^* = P_0 \cup \{v_i\} \cup (\bigcup_{j=1}^{i-1} (C_{k_1}^j) - (v_j, u_j)) \cup \{(v_0, v_1), (u_1, u_2), \ldots, (v_{i-1}, v_i)\}$ (see Fig. 2(a)). If $a = v_i$ and $i - 1$ is odd, then let $P^* = P_0 \cup \{v_i\} \cup (\bigcup_{j=i+1}^{k_2-1} (C_{k_1}^j - (v_j, u_j))) \cup \{(v_0, v_{k_2-1}), (u_{k_2-1}, u_{k_2-2}), \ldots,$

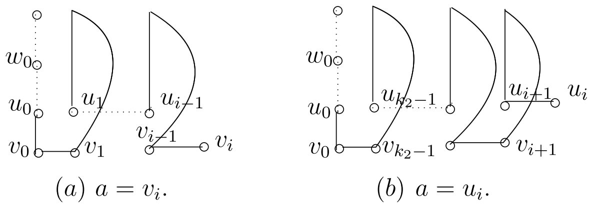

Fig. 2. The fault-free odd path P^* when $i-1$ is even.

$(v_{i+1}, v_i)\}$. If $a = u_i$ and $i-1$ is odd, then let $P^* = P_0 \cup \{u_i\} \cup (\bigcup_{j=1}^{i-1}(C_{k_1}^j - (v_j, u_j))) \cup \{(v_0, v_1), (u_1, u_2), \ldots, (u_{i-1}, u_i)\}$. If $a = u_i$ and $i-1$ is even, then let $P^* = P_0 \cup \{u_i\} \cup (\bigcup_{j=i+1}^{k_2-1}(C_{k_1}^j - (v_j, u_j))) \cup \{(v_0, v_{k_2-1}), (u_{k_2-1}, u_{k_2-2}), \ldots, (u_{i+1}, u_i)\}$ (see Fig. 2(b)). Let M_0 be the matching saturating $C_{k_1}^0 - (V(P_0) \cup F)$ (when $|V(P_0) \cup \{x_0, y_0, z_0\}| = k_1$, $M_0 = \emptyset$). Let M_i be a perfect matching in $C_{k_1}^i - \{w_i, a\}$. Note that P^* is a fault-free odd path. So there exists a perfect matching M^* in P^*. Then $M_0 \cup M^* \cup M_i$ or completing $M_0 \cup M^* \cup M_i$ gives a perfect matching of $T(k_1, k_2) - F$.

Case 1.2.2. $s = 3$.

There exist exactly three even paths P_1, P_2 and P_3 in $C_{k_1}^0 - F$. Assume that w_0 is a terminal vertex of P_k ($k \in \{1, 2, 3\}$) or w_0 is an internal vertex of P_k ($k \in \{1, 2, 3\}$) and $P_k - w_0$ can be partitioned into the set of paths of length one. Without loss of generality, say $k = 1$. Let M_1 be the matching saturating $P_1 - w_0$ (when $|P_1| = 1$, $M_1 = \emptyset$). P_2 and P_3 can be partitioned into the set M_2 of paths of length one plus two single vertices a_0 and b_0 such that a_0 and b_0 are adjacent to one of the fault vertices (say x_0). Let $M_0^* = M_1 \cup M_2$, and let M_i^* be the corresponding matching to M_0^* for $i = 1, 2, \ldots, k_2 - 1$. Let M_{t_0} be the matching saturating $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{a_0, b_0, x_0\}$ (when $k_2 = 3$, $M_{t_0} = \emptyset$). Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{y_0, z_0, w_i\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup (\bigcup_{z \in \{y_0, z_0, w_i\}} M_z) \cup (\bigcup_{t_0 \in \{a_0, b_0, x_0\}} M_{t_0}) \cup \{(a_{k_2-1}, x_{k_2-1}), (a_0, a_1), (b_{k_2-1}, b_0), (x_1, b_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that w_0 is an internal vertex of P_k ($k \in \{1, 2, 3\}$) and $P_k - w_0$ can be partitioned into the set of paths of length one plus two single vertices. Without loss of generality, say $k = 1$. Let $N_{P_1}(w_0) = \{c_0, d_0\}$. Let M_1 be a perfect matching in $P_1 - \{c_0, d_0, w_0\}$. P_2 and P_3 can be partitioned into the set M_2 of paths of length one plus two single vertices a_0 and b_0 such that a_0 and b_0 are adjacent to one of the fault vertices (say x_0). Let M_{t_0} be the matching saturating $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{a_0, b_0, x_0\}$. Let M_{t_i} be the matching saturating $C_{k_2}(t_i) - \{t_{i-1}, t_i, t_{i+1}\}$ for each $t_i \in \{c_i, d_i, w_i\}$. Let $M_0^* = M_1 \cup M_2$. Let M_j^* be the corresponding matching to M_0^* for $j \in \{1, 2, \ldots, k_2 - 1\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{y_0, z_0\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup M_{y_0} \cup M_{z_0} \cup (\bigcup_{t_0 \in \{a_0, b_0, x_0\}} M_{t_0}) \cup (\bigcup_{t_i \in \{c_i, d_i, w_i\}} M_{t_i}) \cup \{(a_{k_2-1}, x_{k_2-1}), (a_0, a_1), (b_{k_2-1}, b_0), (x_1, b_1)\} \cup \{(c_{i-1}, c_i), (c_{i+1}, w_{i+1}), (w_{i-1}, d_{i-1}), (d_i, d_{i+1})\}$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that $w_0 \in \{x_0, y_0, z_0\}$. Without loss of generality, $w_0 = z_0$ and $N_{C_{k_1}^0}(z_0) \cap V(P_1) = \{c_0\}$. Let M_1 be a perfect matching in $P_1 - c_0$. P_2 and P_3 can be partitioned into the set M_2 of paths of length one plus two single vertices a_0 and b_0 such that a_0 and b_0 are adjacent to one of the fault vertices (say x_0). Let M_{t_0} be the matching saturating $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{a_0, b_0, x_0\}$. Let $M_0^* = M_1 \cup M_2$. $C_{k_2}(z_0) - \{z_0, w_i\}$ can be partitioned into the set M_{z_0} of paths of length one plus one single vertex w_j such that $(w_j, w_i) \in E(C_{k_2}(z_0))$ and $w_j \neq z_0$. Let M_{c_0} be a perfect matching in $C_{k_2}(c_j) - c_j$ and let M_j^* be a perfect matching in $C_{k_1}^j - \{y_j, x_j, a_j, b_j\}$. Let M_m^* be the corresponding matching to M_0^* for $m \in \{1, 2, \ldots, k_2 - 1\} \backslash \{j\}$. Let M_{y_0} be a perfect matching in $C_{k_2}(y_0) - y_0$. Then $(\bigcup_{i=0}^{k_2-1} M_i^*) \cup M_{y_0} \cup M_{z_0} \cup M_{c_0} \cup (\bigcup_{t_0 \in \{a_0, b_0, x_0\}} M_{t_0}) \cup \{(a_{k_2-1}, x_{k_2-1}), (a_0, a_1), (b_{k_2-1}, b_0), (x_1, b_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.3. $|F_v \cap V(C_{k_1}^0)| = 2$.

There are two fault vertices in $T(k_1, k_2) - V(C_{k_1}^0)$. Let $0 < i \leqslant j \leqslant k_2 - 1$ and let $a_i \in V(C_{k_1}^i)$ and $b_j \in V(C_{k_1}^j)$ be the two fault vertices. Let $F_v \cap V(C_{k_1}^0) = \{x_0, y_0\}$. We consider five subcases (see Fig. 3).

Case 1.3.1. $a_0 = b_0 \in \{x_0, y_0\}$.

Without loss of generality, say $a_0 = b_0 = x_0$. Assume that $C_{k_2}(x_0) - \{x_0, a_i, b_j\}$ can be partitioned into a set M_{x_0} of paths of length one. If $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set M_0 of paths of length one, then M_0 is a matching saturating $C_{k_1}^0 - \{x_0, y_0\}$. Let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \ldots, k_2 - 1\}$. Let M_{y_0} be a perfect matching in $C_{k_2}(y_0) - y_0$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup M_{x_0} \cup M_{y_0}$ is a perfect matching in $T(k_1, k_2) - F$. If $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set M_0 of paths of length one plus two single vertices u_0 and v_0 such that u_0 and v_0 are adjacent to y_0, then let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \ldots, k_2 - 1\}$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{u_0, y_0, v_0\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{u_0, v_0, x_0, y_0\}} M_z) \cup \{(u_{k_2-1}, y_{k_2-1}), (u_0, u_1), (v_{k_2-1}, v_0), (y_1, v_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that $C_{k_2}(x_0) - \{x_0, a_i, b_j\}$ can be partitioned into a set M_{x_0} of paths of length one plus two single vertices x_m and x_n such that x_m and x_n are adjacent to one vertex in $\{x_0, a_i, b_j\}$ (say a_i). Without loss of generality, let $0 < n < i$.

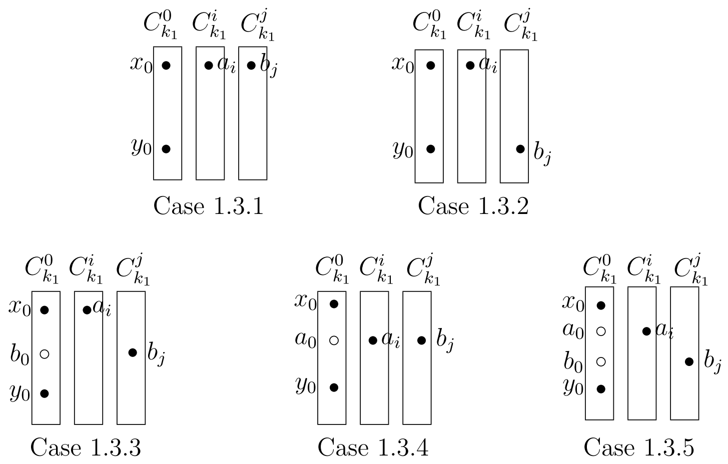

Fig. 3. Configuration of fault vertices in Case 1.3.

If $C^0_{k_1} - \{x_0, y_0\}$ can be partitioned into a set of paths of length one, then there exists $c_m \in N_{C^m_{k_1}}(x_m)$ such that $y_m \notin N_{C^m_{k_1}}(c_m)$. Let $d_i \in V(C^i_{k_1})$ be the neighbour of c_i such that d_i and a_i are distinct. Let M_0 be a perfect matching in $C^0_{k_1} - \{x_0, c_0, d_0, y_0\}$, and let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Let M_{c_i} be a perfect matching in $C_{k_2}(c_i) - \{c_m, c_i, c_n\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{d_i, y_0\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{x_0,y_0,c_i,d_i\}} M_z) \cup \{(x_m, c_m), (x_n, c_n), (c_i, d_i)\}$ is a perfect matching in $T(k_1, k_2) - F$.

If $C^0_{k_1} - \{x_0, y_0\}$ can be partitioned into a set of paths of length one plus two single vertices u_0 and v_0 such that u_0 and v_0 are adjacent to y_0, then we consider two subcases.

Case 1.3.1.1. $x_0 \in N_{C^0_{k_1}}(u_0) \cup N_{C^0_{k_1}}(v_0)$.

Without loss of generality, say $x_0 \in N_{C^0_{k_1}}(u_0)$. Since $k_1 \geqslant 6$, there exist $c_0 \in N_{C^0_{k_1}}(x_0)$ and $d_0 \in N_{C^0_{k_1}}(c_0)$ such that $c_0 \neq u_0$ and $d_0 \neq x_0$. Let M_0 be a perfect matching in $C^0_{k_1} - \{x_0, u_0, v_0, y_0, c_0, d_0\}$, and let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Let M_{c_0} be a perfect matching in $C_{k_2}(c_0) - \{c_m, c_i, c_n\}$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{u_0, y_0, v_0\}$. Let M_{d_0} be a perfect matching in $C_{k_2}(d_0) - d_i$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{x_0,c_0,d_0,y_0,u_0,v_0\}} M_z) \cup \{(x_m, c_m), (x_n, c_n), (c_i, d_i), (u_{k_2-1}, y_{k_2-1}), (u_0, u_1), (v_{k_2-1}, v_0), (y_1, v_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.3.1.2. $x_0 \notin N_{C^0_{k_1}}(u_0) \cup N_{C^0_{k_1}}(v_0)$.

In this case, there exists $c_m \in N_{C^m_{k_1}}(x_m)$ such that $y_m \notin N_{C^m_{k_1}}(c_m)$. Let $d_i \in V(C^i_{k_1})$ be the neighbour of c_i such that d_i and a_i are distinct. Let M_{c_i} be a perfect matching in $C_{k_2}(c_i) - \{c_m, c_i, c_n\}$. Let M_{d_i} be a perfect matching in $C_{k_2}(d_i) - d_i$. $C^0_{k_1} - \{x_0, c_0, d_0, y_0, v_0, u_0\}$ can be partitioned into a set M_0 of paths of length one, and let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{u_0, y_0, v_0\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{x_0,c_i,d_i,y_0,v_0,u_0\}} M_z) \cup \{(x_m, c_m), (x_n, c_n), (c_i, d_i)\} \cup \{(u_{k_2-1}, y_{k_2-1}), (u_0, u_1), (v_{k_2-1}, v_0), (y_1, v_1)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.3.2. $|\{a_0, b_0\} \cap \{x_0, y_0\}| = 2$.

Without loss of generality, say $x_0 = a_0$ and $y_0 = b_0$. $C_{k_2}(x_0) - \{x_0, a_i\}$ can be partitioned into the set M_{x_0} of paths of length one plus one single vertex $x_m \in V(C^m_{k_1})$. $C_{k_2}(y_0) - \{y_0, b_j\}$ can be partitioned into the set M_{y_0} of paths of length one plus one single vertex $y_n \in V(C^n_{k_1})$. If $C^0_{k_1} - \{x_0, y_0\}$ can be partitioned into the set of paths of length one, then let $u_m \in N_{C^m_{k_1}}(x_m)$ and $v_n \in N_{C^n_{k_1}}(y_n)$ such that u_0 is connected to v_0 in $C^0_{k_1} - \{x_0, y_0\}$. If $C^0_{k_1} - \{x_0, y_0\}$ cannot be partitioned into the set of paths of length one, then let $u_m \in N_{C^m_{k_1}}(x_m)$ and $v_n \in N_{C^n_{k_1}}(y_n)$ such that u_0 is disconnected from v_0 in $C^0_{k_1} - \{x_0, y_0\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{u_m, v_n\}$. $C^0_{k_1} - \{x_0, y_0, u_0, v_0\}$ can be partitioned into the set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{x_0,y_0,u_m,v_n\}} M_z) \cup \{(x_m, u_m), (y_n, v_n)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.3.3. $|\{a_0, b_0\} \cap \{x_0, y_0\}| = 1$.

Without loss of generality, say $x_0 = a_0$. Assume that $C^0_{k_1} - \{x_0, y_0, b_0\}$ can be partitioned into a set of paths of length one plus some single vertices. Denote by s the number of these single vertices. It is easy to see that $1 \leqslant s \leqslant 3$. Since $|V(C^0_{k_1}) \backslash \{x_0, y_0, b_0\}|$ is odd, $s \neq 2$.

Case 1.3.3.1. $N_{C^0_{k_1}}(x_0) \neq \{b_0, y_0\}$.

If $s=1$ and $C_{k_1}^0-\{x_0,b_0,y_0\}$ can be partitioned into a set of paths of length one plus one single vertex c_0 such that c_0 is adjacent to x_0, then there is a matching M_0 saturating $C_{k_1}^0-\{x_0,y_0,b_0,c_0\}$. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. $C_{k_2}(x_0)-\{x_0,a_i\}$ can be partitioned into a set M_{x_0} of paths of length one plus one single vertex x_m. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{c_m,b_j,y_0\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,y_0,c_m,b_j\}}M_z)\cup\{(x_m,c_m)\}$ is a perfect matching in $T(k_1,k_2)-F$.

If $s=1$, $y_0\in N_{C_{k_1}^0}(x_0)$ and $C_{k_1}^0-\{x_0,b_0,y_0\}$ cannot be partitioned into a set of paths of length one plus one single vertex which is adjacent to x_0, then there exists $c_0\in N_{C_{k_1}^0}(y_0)$ such that $c_0\neq x_0$. $C_{k_2}(x_0)-\{x_0,a_i\}$ can be partitioned into a set M_{x_0} of paths of length one plus one single vertex x_m. Let M_{y_0} be a perfect matching in $C_{k_2}(y_0)-\{y_0,y_i,y_m\}$. Let M_0 be a perfect matching in $C_{k_1}^0-\{x_0,y_0,c_0,b_0\}$ and let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{c_i,b_j\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,y_0,c_i,b_j\}}M_z)\cup\{(x_m,y_m),(y_i,c_i)\}$ is a perfect matching in $T(k_1,k_2)-F$.

If $s=1$, $y_0\notin N_{C_{k_1}^0}(x_0)$ and $C_{k_1}^0-\{x_0,b_0,y_0\}$ cannot be partitioned into a set of paths of length one plus one single vertex which is adjacent to x_0, then there exists $c_0\in N_{C_{k_1}^0}(x_0)$ such that c_0 is disconnected from b_0 in $C_{k_1}^0-\{x_0,y_0\}$. If $s=3$, then let $c_0\in N_{C_{k_1}^0}(x_0)$ such that c_0 is connected to b_0 in $C_{k_1}^0-\{x_0,y_0\}$. $C_{k_1}^0-\{x_0,y_0,b_0,c_0\}$ can be partitioned into a set M_0 of paths of length one plus two single vertices u_0 and v_0 such that u_0 and v_0 are adjacent to y_0. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. $C_{k_2}(x_0)-\{x_0,a_i\}$ can be partitioned into a set M_{x_0} of paths of length one plus one single vertex x_m. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{c_m,b_j\}$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0)-\{t_0,t_{k_2-1},t_1\}$ for each $t_0\in\{u_0,y_0,v_0\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,c_m,b_j,u_0,y_0,v_0\}}M_z)\cup\{(u_{k_2-1},y_{k_2-1}),(u_0,u_1),(v_{k_2-1},v_0),(y_1,v_1),(x_m,c_m)\}$ is a perfect matching in $T(k_1,k_2)-F$.

Case 1.3.3.2. $N_{C_{k_1}^0}(x_0)=\{b_0,y_0\}$.

$C_{k_2}(x_0)-\{x_0,a_i\}$ can be partitioned into a set M_{x_0} of paths of length one plus one single vertex x_m. Similarly, $C_{k_2}(y_0)-\{y_0,y_i\}$ can be partitioned into a set M_{y_0} of paths of length one plus one single vertex y_m. Let $d_i\in V(C_{k_1}^i)$ be the neighbour of y_i such that d_i and a_i are distinct. $C_{k_1}^0-\{x_0,b_0,y_0,d_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{d_i,b_j\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,y_0,d_i,b_j\}}M_z)\cup\{(x_m,y_m),(d_i,y_i)\}$ is a perfect matching in $T(k_1,k_2)-F$.

Case 1.3.4. $|\{a_0,b_0\}\cap\{x_0,y_0\}|=0$ and $a_0=b_0$.

$C_{k_1}^0-\{x_0,y_0\}$ is divided into two paths P_1 and P_2. Without loss of generality, say $a_0\in V(P_1)$. If $|P_1|=1$ and $\{i,j\}=\{1,k_2-1\}$, then F is a trivial strong matching preclusion set. If $|P_1|=1$ and $\{i,j\}\neq\{1,k_2-1\}$, then there exists $a_m\in V(C_{k_1}^m)$ $(m\neq 0)$ such that $C_{k_2}(a_0)-\{a_i,a_j,a_m\}$ can be partitioned into a set M_{a_0} of paths of length one. Since $|P_2|$ is odd and $k_1\geqslant 6$, there exists $c_0\in N_{C_{k_1}^0}(y_0)$ such that $c_0\neq a_0$. $C_{k_2}(y_0)-\{y_0,y_m\}$ can be partitioned into a set M_{y_0} of paths of length one plus one single vertex y_n. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{x_0,c_n\}$. $C_{k_1}^0-\{x_0,a_0,y_0,c_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,a_0,y_0,c_n\}}M_z)\cup\{(a_m,y_m),(y_n,c_n)\}$ is a perfect matching in $T(k_1,k_2)-F$.

If $|P_1|\geqslant 3$ and $|P_1|$ is even, then $|P_2|$ is even. There exists $c_0\in V(P_1)$ such that $(c_0,a_0)\in E(P_1)$ and $C_{k_1}^0-\{x_0,a_0,y_0,c_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. $C_{k_2}(a_0)-\{a_i,b_j\}$ can be partitioned into a set M_{a_0} of paths of length one plus one single vertex a_m. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{x_0,y_0,c_m\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,y_0,a_0,c_m\}}M_z)\cup\{(a_m,c_m)\}$ is a perfect matching in $T(k_1,k_2)-F$.

If $|P_1|\geqslant 3$ and $|P_1|$ is odd, then $|P_2|$ is odd. There exists $c_0\in V(P_1)$ such that $(c_0,a_0)\in E(P_1)$ and $C_{k_1}^0-\{x_0,a_0,y_0,c_0\}$ can be partitioned into a set M_0 of paths of length one plus two single vertices u_0 and v_0 satisfying u_0 and v_0 are adjacent to y_0. Let M_i be the corresponding matching to M_0 for $i\in\{1,2,\dots,k_2-1\}$. $C_{k_2}(a_0)-\{a_i,b_j\}$ can be partitioned into a set M_{a_0} of paths of length one plus one single vertex a_m. Let M_z be a perfect matching in $C_{k_2}(z)-z$ for each $z\in\{x_0,c_m\}$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0)-\{t_0,t_{k_2-1},t_1\}$ for each $t_0\in\{u_0,y_0,v_0\}$. Then $(\bigcup_{i=0}^{k_2-1}M_i)\cup(\bigcup_{z\in\{x_0,c_m,a_0,u_0,y_0,v_0\}}M_z)\cup\{(u_{k_2-1},y_{k_2-1}),(u_0,u_1),(v_{k_2-1},v_0),(y_1,v_1),(a_m,c_m)\}$ is a perfect matching in $T(k_1,k_2)-F$.

Case 1.3.5. $|\{a_0,b_0\}\cap\{x_0,y_0\}|=0$ and $a_0\neq b_0$.

Similarly to the proof of Case 1.1, we can obtain a perfect matching in $T(k_1,k_2)-F$.

Case 1.4. $|F_v\cap V(C_{k_1}^0)|=1$.

For $0<j<m<n\leqslant k_2-1$, let $a_0\in V(C_{k_1}^0)$, $b_j\in V(C_{k_1}^j)$, $c_m\in V(C_{k_1}^m)$ and $d_n\in V(C_{k_1}^n)$ be the fault vertices. We consider five subcases.

Case 1.4.1. $a_0=b_0=c_0=d_0$.

$C_{k_2}(a_0)-F_v$ can be partitioned into a set of paths of length one plus some single vertices. Denote by s the number of these single vertices. It is easy to see that $0<s\leqslant 4$. Since $|V(C_{k_2}(a_0))\backslash F_v|$ is odd, s is odd. We consider two subcases.

Case 1.4.1.1. $s=1$.

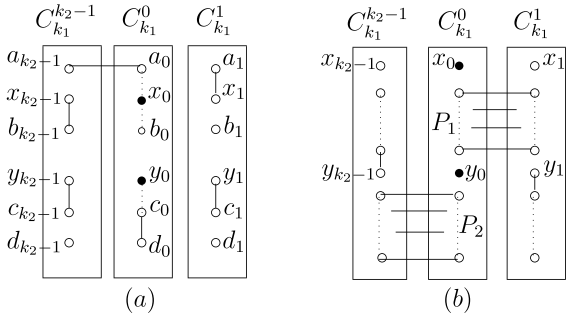

Fig. 4. $F_v \cap V(C_{k_1}^0) = \{x_0, y_0\}$.

$C_{k_2}(a_0) - F_v$ can be partitioned into a set M_{a_0} of paths of length one plus one single vertex a_i. Let $u_i \in N_{C_{k_1}^i}(a_i)$. $C_{k_1}^0 - \{a_0, u_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \ldots, k_2 - 1\}$. Let M_{u_i} be a perfect matching in $C_{k_2}(u_i) - u_i$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{a_0, u_i\}} M_z) \cup \{(a_i, u_i)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.4.1.2. $s = 3$.

$C_{k_2}(a_0) - F_v$ can be partitioned into a set M_{a_0} of paths of length one plus three single vertices a_i, a_p and a_q such that a_p and a_q are adjacent to one of the fault vertices (say c_m). Let $u_i, w_i \in N_{C_{k_1}^i}(a_i)$ such that $w_i \neq u_i$. Let $v_m \in V(C_{k_1}^m)$ be the neighbour of w_m such that v_m and c_m are distinct. $C_{k_1}^0 - \{a_0, u_0, w_0, v_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \ldots, k_2 - 1\}$. $C_{k_2}(w_p) - \{w_p, w_m, w_q\}$ can be partitioned into a set M_{w_p} of paths of length one. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{v_m, u_i\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{a_0, w_p, v_m, u_i\}} M_z) \cup \{(a_i, u_i), (a_p, w_p), (a_q, w_q), (w_m, v_m)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 1.4.2. There are exactly three fault vertices in some $V(C_{k_2}(x))$, where $x \in \{a_0, b_j, c_m, d_n\}$.

Without loss of generality, say $a_0 = b_0 = c_0$. Similarly to the proof of Case 1.3.1, we can obtain a perfect matching in $T(k_1, k_2) - F$.

Case 1.4.3. Exactly two fault vertices are in $V(C_{k_2}(x))$ for some $x \in \{a_0, b_j, c_m, d_n\}$ and the other two fault vertices are in $V(C_{k_2}(y))$ for some $y \neq x$ and $y \in \{a_0, b_j, c_m, d_n\}$.

Without loss of generality, say $a_0 = b_0$ and $c_0 = d_0$. Similarly to the proof of Case 1.3.2, we can obtain a perfect matching in $T(k_1, k_2) - F$.

Case 1.4.4. Exactly two fault vertices are in $V(C_{k_2}(x))$ for some $x \in \{a_0, b_j, c_m, d_n\}$ and the other two fault vertices are not in $V(C_{k_2}(y))$ for some $y \neq x$ and $y \in \{a_0, b_j, c_m, d_n\}$.

Without loss of generality, say $a_0 = b_0$ and $c_0 \neq d_0$. Similarly to the proof of Case 1.3.3, we can obtain a perfect matching in $T(k_1, k_2) - F$.

Case 1.4.5. a_0, b_0, c_0 and d_0 are four distinct vertices in $V(C_{k_1}^0)$.

Similarly to the proof of Case 1.1, we can obtain a perfect matching in $T(k_1, k_2) - F$.

Case 2. $|F_v| = 3$.

In this case, $|F_e| \leqslant 1$. If there is no fault edges or the fault edge is incident to one of the vertices in F_v, then let $v^* \in V(T(k_1, k_2)) \setminus F_v$ be a vertex such that $F_v \cup \{v^*\}$ is not a trivial strong matching preclusion set. If the fault edge is not incident to any vertex in F_v, then let v^* be the vertex such that the fault edge is incident to v^* and $F_v \cup \{v^*\}$ is not a trivial strong matching preclusion set. Let $F^* = F_v \cup \{v^*\}$. By the proof of Case 1, there exists a perfect matching M in $T(k_1, k_2) - F^*$. Note that M saturates all the vertices in $T(k_1, k_2) - F$ except the vertex v^*. Thus, M gives an almost perfect matching of $T(k_1, k_2) - F$.

Case 3. $|F_v| = 2$.

It is enough to consider the case when there is no fault edge which is incident to any fault vertex.

Case 3.1. $|F_v \cap V(C_{k_1}^0)| = 2$.

Let x_0 and y_0 be the fault vertices. Since $|F| = |F_v \cup F_e| \leqslant 4$ and $|F_v| = 2$, $|F_e| \leqslant 2$. We consider three subcases.

Case 3.1.1. There are no fault edges in $C_{k_1}^0$.

Assume that $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set M_0 of paths of length one. Note that there exists a perfect matching in each copy of C_{k_1} with at most one fault edge. If two fault edges are in $C_{k_1}^i$ for some $i \in \{1, 2, \ldots, k_2 - 1\}$, then there exists $j \in \{i + 1, i - 1\}$ such that $j \in \{1, 2, \ldots, k_2 - 1\}$ and $M_0 \cup M_{i,j}$ or completing $M_0 \cup M_{i,j}$ gives a perfect matching of $T(k_1, k_2) - F$. Otherwise, completing M_0 gives a perfect matching of $T(k_1, k_2) - F$.

Assume that $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one plus two single vertices. $C_{k_1}^0 - \{x_0, y_0\}$ is divided into two even paths P_1 and P_2. Without loss of generality, $|P_1| \leqslant |P_2|$.

Case 3.1.1.1. There is one fault edge in $C_{k_2}(x_0)$ and there is one fault edge in $C_{k_2}(y_0)$ (see Fig. 4(a)).

Let $N_{C^0_{k_1}}(x_0) = \{a_0, b_0\}$. Let $c_0 \in V(C^0_{k_1})$ be the neighbour of y_0 such that $c_0 \notin \{a_0, b_0\}$. Let $d_0 \in V(C^0_{k_1})$ be the neighbour of c_0 such that d_0 and y_0 are distinct. $C^0_{k_1} - \{x_0, y_0, a_0, b_0, c_0, d_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{d_0, b_{k_2-1}\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup M_{d_0} \cup M_{b_{k_2-1}} \cup (\bigcup_{j=1}^{k_2-2}\{(x_j, a_j), (y_j, c_j)\}) \cup \{(a_0, a_{k_2-1}), (x_{k_2-1}, b_{k_2-1}), (y_{k_2-1}, c_{k_2-1}), (c_0, d_0)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 3.1.1.2. Either $C_{k_2}(x_0)$ contains no fault edges or $C_{k_2}(y_0)$ contains no fault edges.

Without loss of generality, $C_{k_2}(x_0)$ contains no fault edges. Let $M_0 = \{(u_0, u_1)\colon u_0 \in V(P_1)\}$ and $M'_0 = \{(v_0, v_{k_2-1})\colon v_0 \in V(P_1)\}$. If $|P_1| = 1$ and $M_0 \cup M'_0$ contains two fault edges, then F is a trivial strong matching preclusion set. Next, we consider the condition that F is not a trivial strong matching preclusion set.

Assume that the even cycle $C = (a_0, a_1, y_1, b_1, b_0, b_{k_2-1}, y_{k_2-1}, a_{k_2-1}, a_0)$ contains at most one fault edge, where $\{a_0, b_0\} = N_{C^0_{k_1}}(y_0)$. Then there exists a perfect matching M' in C. Let M_0 be a perfect matching in $C^0_{k_1} - \{x_0, y_0, a_0, b_0\}$ and let M_i be the corresponding matching to M_0 for $i \in \{1, 2, \dots, k_2 - 1\}$. Let M_{x_0} be a perfect matching in $C_{k_2}(x_0) - x_0$. Let M_{t_0} be a perfect matching in $C_{k_2}(t_0) - \{t_0, t_{k_2-1}, t_1\}$ for each $t_0 \in \{y_0, a_0, b_0\}$. Let $M^* = (\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{x_0,y_0,a_0,b_0\}} M_z) \cup M'$. If $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z\in\{y_0,a_0,b_0\}} M_z)$ contains no fault edges, then M^* is a perfect matching in $T(k_1, k_2) - F$. If there is one fault edge (u_j, v_j) in $\bigcup_{i=0}^{k_2-1} M_i$ for $j \in \{0, 1, \dots, k_2 - 1\}$ and there is one fault edge (a_l, a_{l+1}) in $\bigcup_{z\in\{y_0,a_0,b_0\}} M_z$, then $M^* \cup \{(u_j, u_{j+1}), (v_j, v_{j+1}), (a_l, y_l), (a_{l+1}, y_{l+1})\} \backslash \{(u_j, v_j), (u_{j+1}, v_{j+1}), (a_l, a_{l+1}), (y_l, y_{l+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. If there are at most two fault edges (u_j, v_j) and (w_l, z_l) in $\bigcup_{i=0}^{k_2-1} M_i$ for $j, l \in \{0, 1, \dots, k_2 - 1\}$ and there are no fault edges in $\bigcup_{z\in\{y_0,a_0,b_0\}} M_z$, then either $\{w_l, z_l\} \subseteq N_{C_{k_2}(u_j)}(u_j) \cup N_{C_{k_2}(v_j)}(v_j)$ (say $u_{j+1} = w_l$ and $v_{j+1} = z_l$) or there exist $j^* \in \{j-1, j+1\}$ and $l^* \in \{l-1, l+1\}$ such that $(u_j, u_{j^*}), (v_j, v_{j^*}), (w_l, w_{l^*}), (z_l, z_{l^*})$ are not fault edges. Thus, $M^* \cup \{(u_j, w_l), (v_j, z_l)\} \backslash \{(u_j, v_j), (w_l, z_l)\}$ or $M^* \cup \{(u_j, u_{j^*}), (v_j, v_{j^*}), (w_l, w_{l^*}), (z_l, z_{l^*})\} \backslash \{(u_j, v_j), (u_{j^*}, v_{j^*}), (w_l, z_l), (w_{l^*}, z_{l^*})\}$ is a perfect matching in $T(k_1, k_2) - F$. If there are two fault edges $(u_m, u_{m+1}), (v_n, v_{n+1})$ in $\bigcup_{z\in\{y_0,a_0,b_0\}} M_z$, then we consider the following three subcases: (1) when $m = n$ and $(u_m, v_n) \in E(C^m_{k_1})$, $M^* \cup \{(u_m, v_n), (u_{m+1}, v_{n+1})\} \backslash \{(u_m, u_{m+1}), (v_n, v_{n+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. (2) When $m = n$ and $(u_m, v_n) \notin E(C^m_{k_1})$, without loss of generality, say $x_m \notin N_{C^m_{k_1}}(u_m)$, there exist $w_m \notin \{y_m, a_m, b_m, x_m\}$ and $z_n \in \{y_n, a_n, b_n\}$ such that $(u_m, w_m), (u_{m+1}, w_{m+1}), (v_n, z_n), (v_{n+1}, z_{n+1}), (o_m, o_{m+1})$ are not fault edges, where $o_m \in N_{C^m_{k_1}}(w_m)$ and $o_m \neq u_m$, $M^* \cup \{(u_m, w_m), (u_{m+1}, w_{m+1}), (v_n, z_n), (v_{n+1}, z_{n+1}), (o_m, o_{m+1})\} \backslash \{(u_m, u_{m+1}), (v_n, v_{n+1}), (z_n, z_{n+1}), (w_m, o_m), (w_{m+1}, o_{m+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. (3) When there exist $w_m \in \{y_m, a_m, b_m\}$ and $z_n \in \{y_n, a_n, b_n\}$ such that $(u_m, w_m), (u_{m+1}, w_{m+1}), (v_n, z_n), (v_{n+1}, z_{n+1})$ are not fault edges, $M^* \cup \{(u_m, w_m), (u_{m+1}, w_{m+1}), (v_n, z_n), (v_{n+1}, z_{n+1})\} \backslash \{(u_m, u_{m+1}), (v_n, v_{n+1}), (w_m, w_{m+1}), (z_n, z_{n+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. If there is exactly one fault edge (u_m, u_{m+1}) in $\bigcup_{z\in\{y_0,a_0,b_0\}} M_z$, then we consider the following three subcases: (1) when there exists $w_m \in \{y_m, a_m, b_m\}$ such that $(u_m, w_m), (u_{m+1}, w_{m+1})$ are not fault edges, $M^* \cup \{(u_m, w_m), (u_{m+1}, w_{m+1})\} \backslash \{(u_m, u_{m+1}), (w_m, w_{m+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. (2) When there exists $w_m \notin \{y_m, a_m, b_m, x_m\}$ such that $(u_m, w_m), (u_{m+1}, w_{m+1}), (o_m, o_{m+1})$ are not fault edges, where $o_m \in N_{C^m_{k_1}}(w_m)$ and $o_m \neq u_m$, $M^* \cup \{(u_m, w_m), (u_{m+1}, w_{m+1}), (o_m, o_{m+1})\} \backslash \{(u_m, u_{m+1}), (w_m, o_m), (w_{m+1}, o_{m+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. (3) When $N_{C^m_{k_1}}(u_m) = \{x_m, y_m\}$ and the other fault edge is incident to y_m or y_{m+1} (say y_m), there are no fault edges in C. Let M'_1 be the perfect matching of C such that $(u_1, y_1) \in M'_1$. If $m = 3$, then $(M^* \backslash M') \cup M'_1 \cup \{(u_1, u_2), (y_1, y_2), (u_{m+1}, y_{m+1})\} \backslash \{(u_1, y_1), (u_m, u_{m+1}), (y_m, y_{m+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. If $m > 3$, then $M^* \cup \{(u_{m-2}, y_{m-2}), (u_{m-1}, u_m), (y_{m-1}, y_m), (u_{m+1}, y_{m+1})\} \backslash \{(u_{m-2}, u_{m-1}), (y_{m-2}, y_{m-1}), (u_m, u_{m+1}), (y_m, y_{m+1})\}$ is a perfect matching in $T(k_1, k_2) - F$.

Assume that the even cycle $(a_0, a_1, y_1, b_1, b_0, b_{k_2-1}, y_{k_2-1}, a_{k_2-1}, a_0)$ contains two fault edges, where $\{a_0, b_0\} = N_{C^0_{k_1}}(y_0)$. Then $C_{k_2}(y_0)$ contains no fault edges. Since F is not a trivial strong matching preclusion set, the even cycle $(c_0, c_1, x_1, d_1, d_0, d_{k_2-1}, x_{k_2-1}, c_{k_2-1}, c_0)$ contains at most one fault edge, where $\{c_0, d_0\} = N_{C^0_{k_1}}(x_0)$. Similarly to the proof of the above discussion, we can obtain a perfect matching in $T(k_1, k_2) - F$.

Case 3.1.2. There is exactly one fault edge $e_1 = (u_0, v_0)$ in $C^0_{k_1}$.

Assume that $C^0_{k_1} - \{x_0, y_0\}$ can be partitioned into a set of paths of length one. Note that there exists a perfect matching in each copy of C_{k_1} with at most one fault edge. If $C^0_{k_1} - \{x_0, y_0\} - e_1$ can be partitioned into a set M_0 of paths of length one, then completing M_0 gives a perfect matching of $T(k_1, k_2) - F$. Otherwise, there exists $j \in \{1, k_2 - 1\}$ such that neither (u_0, u_j) nor (v_0, v_j) is faulty and $C^j_{k_1}$ contains no fault edges. Now, $C^0_{k_1} - \{x_0, y_0, u_0, v_0\}$ can be partitioned into a set M_0 of paths of length one and $C^j_{k_1} - \{u_j, v_j\}$ can be partitioned into a set M_j of paths of length one. Then completing $M_0 \cup M_j \cup \{(u_0, u_j), (v_0, v_j)\}$ gives a perfect matching of $T(k_1, k_2) - F$.

Assume that $C^0_{k_1} - \{x_0, y_0\}$ can be partitioned into a set of paths of length one plus two single vertices. $C^0_{k_1} - \{x_0, y_0\}$ is divided into two even paths P_1 and P_2. Let $M_0 = \{(u_0, u_1)\colon u_0 \in V(P_1)\} \cup \{(v_0, v_{k_2-1})\colon v_0 \in V(P_2)\}$ and $M'_0 = \{(u_0, u_1)\colon u_0 \in V(P_2)\} \cup \{(v_0, v_{k_2-1})\colon v_0 \in V(P_1)\}$. Let $V_1 = \{w_1\colon w_1 \in V(C^1_{k_1})$ and $w_0 \in V(P_1)\}$ and $V_{k_2-1} = \{w_{k_2-1}\colon w_{k_2-1} \in V(C^{k_2-1}_{k_1})$ and $w_0 \in V(P_2)\}$. Then $C^1_{k_1} - (V_1 \cup \{x_1\})$ can be partitioned into a set M_1 of paths of length one and $C^{k_2-1}_{k_1} - (V_{k_2-1} \cup \{x_{k_2-1}\})$

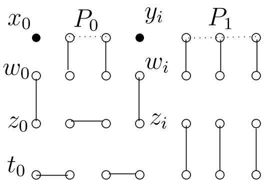

Fig. 5. $F_v \cap V(C_{k_1}^0) = \{x_0\}$ and $x_0 = y_0$.

can be partitioned into a set M_{k_2-1} of paths of length one. Let $V_1' = \{w_1\colon\ w_1 \in V(C_{k_1}^1)$ and $w_0 \in V(P_2)\}$ and $V_{k_2-1}' = \{w_{k_2-1}\colon\ w_{k_2-1} \in V(C_{k_1}^{k_2-1})$ and $w_0 \in V(P_1)\}$. Then $C_{k_1}^1 - (V_1' \cup \{x_1\})$ can be partitioned into a set M_1' of paths of length one and $C_{k_1}^{k_2-1} - (V_{k_2-1}' \cup \{x_{k_2-1}\})$ can be partitioned into a set M_{k_2-1}' of paths of length one. Since there is at most one fault edge in $T(k_1,k_2) - V(C_{k_1}^0)$, either M_0, M_1 and M_{k_2-1} contain no fault edges or M_0', M_1' and M_{k_2-1}' contain no fault edges. Without loss of generality, M_0, M_1 and M_{k_2-1} contain no fault edges (see Fig. 4(b)). Similarly, either $C_{k_2}(x_0)$ or $C_{k_2}(y_0)$ contains no fault edges. Without loss of generality, $C_{k_2}(x_0)$ contains no fault edges. Let M_{x_0} be a perfect matching in $C_{k_2}(x_0) - x_0$. If there is no fault cross edges in $M_{2j,2j+1}$ for any $j \in \{1,\dots,\frac{k_2-1}{2}-1\}$, then $(\bigcup_{i=1}^{\frac{k_2-1}{2}-1}(M_{2i,2i+1}\backslash\{(x_{2i},x_{2i+1})\})) \cup M_0 \cup M_1 \cup M_{k_2-1} \cup M_{x_0}$ is a perfect matching in $T(k_1,k_2) - F$. If there is one fault cross edge (c_{2j},c_{2j+1}) in $M_{2j,2j+1}$ for some $j \in \{1,\dots,\frac{k_2-1}{2}-1\}$, then there exists $d_{2j} \in V(C_{k_1}^{2j})$ such that d_{2j} is a neighbour of c_{2j} in $C_{k_1}^{2j} - x_{2j}$. Thus $(\bigcup_{i=1}^{\frac{k_2-1}{2}-1}(M_{2i,2i+1}\backslash\{(x_{2i},x_{2i+1})\})) \cup M_0 \cup M_1 \cup M_{k_2-1} \cup M_{x_0} \cup \{(c_{2j},d_{2j}),(c_{2j+1},d_{2j+1})\}\backslash\{(c_{2j},c_{2j+1}),(d_{2j},d_{2j+1})\}$ is a perfect matching in $T(k_1,k_2) - F$.

Case 3.1.3. There are two fault edges $e_1 = (u_0,v_0)$ and $e_2 = (w_0,z_0)$ in $C_{k_1}^0$.

Assume that $C_{k_1}^0 - \{x_0,y_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_i be the corresponding matching to M_0 for $i \in \{2,\dots,k_2-1\}$. Let M_t be a perfect matching in $C_{k_2}(t) - t$ for each $t \in \{x_0,y_0\}$. Then $(\bigcup_{i=2}^{k_2-1} M_i) \cup M_{x_0} \cup M_{y_0} \cup M_{0,1}\backslash\{(x_0,x_1),(y_0,y_1)\}$ is a perfect matching in $T(k_1,k_2) - F$.

Assume that $C_{k_1}^0 - \{x_0,y_0\}$ can be partitioned into a set of paths of length one plus two single vertices. $C_{k_1}^0 - \{x_0,y_0\}$ is divided into two even paths P_1 and P_2. Let $M_0 = \{(u_0,u_1)\colon\ u_0 \in V(P_1)\} \cup \{(v_0,v_{k_2-1})\colon\ v_0 \in V(P_2)\}$. Let $V_1 = \{w_1\colon\ w_1 \in V(C_{k_1}^1)$ and $w_0 \in V(P_1)\}$ and $V_{k_2-1} = \{w_{k_2-1}\colon\ w_{k_2-1} \in V(C_{k_1}^{k_2-1})$ and $w_0 \in V(P_2)\}$. Then $C_{k_1}^1 - (V_1 \cup \{x_1\})$ can be partitioned into a set M_1 of paths of length one and $C_{k_1}^{k_2-1} - (V_{k_2-1} \cup \{x_{k_2-1}\})$ can be partitioned into a set M_{k_2-1} of paths of length one. Let M_{x_0} be a perfect matching in $C_{k_2}(x_0) - x_0$. Thus $(\bigcup_{i=1}^{\frac{k_2-1}{2}-1}(M_{2i,2i+1}\backslash\{(x_{2i},x_{2i+1})\})) \cup M_0 \cup M_1 \cup M_{k_2-1} \cup M_{x_0}$ is a perfect matching in $T(k_1,k_2) - F$.

Case 3.2. $|F_v \cap V(C_{k_1}^0)| = 1$.

Let $x_0 \in V(C_{k_1}^0)$ and $y_i \in V(C_{k_1}^i)$ be the fault vertices, where $0 < i \leqslant k_2 - 1$. We consider two subcases.

Case 3.2.1. $y_0 = x_0$.

Case 3.2.1.1. There is at least one fault edge in $C_{k_2}(x_0)$.

Let $C_{k_1}^0 = (x_0,w_0,z_0,t_0,\dots,c_0,b_0,a_0,x_0)$. $C_{k_2}(x_0) - \{x_0,y_i\}$ can be divided into an odd path P_0 and an even path P_1. Let $V_{s_0} = \{s_j\colon\ s_j \in C_{k_2}(s_0)$ and $x_j \in V(P_1)\}$ for each $s_0 \in \{z_0,t_0,b_0,c_0\}$. Let M_{z_0} and M_{b_0} be the perfect matchings in $C_{k_2}(z_0) - (V_{z_0} \cup \{z_0,z_i\})$ and $C_{k_2}(b_0) - (V_{b_0} \cup \{b_0,b_i\})$, respectively. Let M_{t_0} and M_{c_0} be the perfect matchings in $C_{k_2}(t_0) - V_{t_0}$ and $C_{k_2}(c_0) - V_{c_0}$, respectively. Note that $k_1 \geqslant 6$ and there is at most one fault edge in $T(k_1,k_2) - V(C_{k_2}(x_0))$. If $t_0 = c_0$ and $M_{t_0} \cap F = \emptyset$ or $t_0 \neq c_0$, then either $(\bigcup_{j\in\{1,\dots,k_2-1\}\backslash\{i\}}\{(x_j,w_j)\}) \cup (\bigcup_{z_j\in V_{z_0}}\{(z_j,t_j)\}) \cup M_{z_0} \cup M_{t_0} \cup \{(w_0,z_0),(w_i,z_i)\}$ or $(\bigcup_{j\in\{1,\dots,k_2-1\}\backslash\{i\}}\{(x_j,a_j)\}) \cup (\bigcup_{b_j\in V_{b_0}}\{(b_j,c_j)\}) \cup M_{b_0} \cup M_{c_0} \cup \{(a_0,b_0),(a_i,b_i)\}$ contains no fault edges. Without loss of generality, $(\bigcup_{j\in\{1,\dots,k_2-1\}\backslash\{i\}}\{(x_j,w_j)\}) \cup (\bigcup_{z_j\in V_{z_0}}\{(z_j,t_j)\}) \cup M_{z_0} \cup M_{t_0} \cup \{(w_0,z_0),(w_i,z_i)\}$ contains no fault edges (see Fig. 5).

Let M_0 be a perfect matching in $C_{k_1}^0 - \{x_0,w_0,z_0,t_0\}$ and let M_i be the corresponding matching to M_0 for $i \in \{1,\dots,k_2-1\}$. Let $M^* = (\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{j\in\{1,\dots,k_2-1\}\backslash\{i\}}\{(x_j,w_j)\}) \cup (\bigcup_{z_j\in V_{z_0}}\{(z_j,t_j)\}) \cup M_{z_0} \cup M_{t_0} \cup \{(w_0,z_0),(w_i,z_i)\}$. When there is no fault edge in $\bigcup_{i=0}^{k_2-1} M_i$, M^* is a perfect matching in $T(k_1,k_2) - F$. When $(g_j,h_j) \in \bigcup_{i=0}^{k_2-1} M_i$ is the other fault edge, $M^* \cup \{(g_j,g_{j+1}),(h_j,h_{j+1})\}\backslash\{(g_j,h_j),(g_{j+1},h_{j+1})\}$ is a perfect matching in $T(k_1,k_2) - F$. If $t_0 = c_0$ and $M_{t_0} \cap F \neq \emptyset$, then $|M_{t_0} \cap F| = 1$. Let $(t_{j_1},t_{j_2}) \in M_{t_0}$ be the fault edge. $M^* \cup \{(t_{j_1},b_{j_1}),(t_{j_2},b_{j_2}),(a_{j_1},a_{j_2})\}\backslash\{(t_{j_1},t_{j_2}),(b_{j_1},a_{j_1}),(b_{j_2},a_{j_2})\}$ is a perfect matching in $T(k_1,k_2) - F$.

Case 3.2.1.2. There is no fault edge in $C_{k_2}(x_0)$.

Suppose that F is not a trivial strong matching preclusion set, then $C_{k_2}(x_0) - \{x_0,y_i\}$ can be partitioned into a set M_{x_0} of paths of length one plus one single vertex x_{i^*} such that (x_{i^*},w_{i^*}) is not faulty, where $w_{i^*} \in N_{C_{k_1}^{i^*}}(x_{i^*})$. Let M_{w_0} be a perfect matching in $C_{k_2}(w_0) - w_{i^*}$. Let M_0 be a perfect matching in $C_{k_1}^0 - \{x_0,w_0\}$ and let M_i be the corresponding matching to M_0 for $i \in \{1,\dots,k_2-1\}$. Let $M^* = (\bigcup_{i=0}^{k_2-1} M_i) \cup M_{x_0} \cup M_{w_0} \cup \{(x_{i^*},w_{i^*})\}$. We consider the following

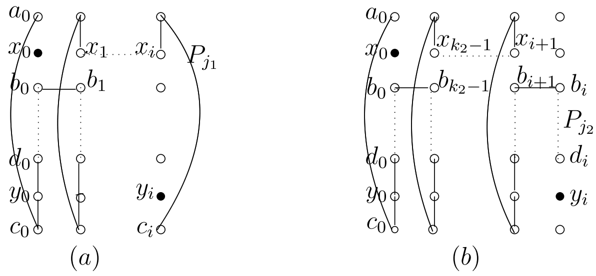

Fig. 6. The odd path P^* when i is even.

four subcases: (1) assume that neither $\bigcup_{i=0}^{k_2-1} M_i$ nor M_{w_0} contains no fault edges, then M^* is a perfect matching in $T(k_1,k_2)-F$. (2) Assume that there are at most two fault edges (a_{j_1},b_{j_1}) and (c_{j_2},d_{j_2}) in $\bigcup_{i=0}^{k_2-1} M_i$ and there is no fault edge in M_{w_0}. If $\{c_{j_2},d_{j_2}\}\subseteq N_{C_{k_2}(a_{j_1})}(a_{j_1})\cup N_{C_{k_2}(b_{j_1})}(b_{j_1})$, without loss of generality, say $a=c$ and $b=d$, then $M^*\cup\{(a_{j_1},c_{j_2}),(b_{j_1},d_{j_2})\}\backslash\{(a_{j_1},b_{j_1}),(c_{j_2},d_{j_2})\}$ is a perfect matching in $T(k_1,k_2)-F$. Otherwise, there exist $j_1^*\in\{j_1+1,j_1-1\}$ and $j_2^*\in\{j_2+1,j_2-1\}$ such that $j_1^*\neq j_2^*$. Then $M^*\cup\{(a_{j_1},a_{j_1^*}),(b_{j_1},b_{j_1^*}),(c_{j_2},c_{j_2^*}),(d_{j_2},d_{j_2^*})\}\backslash\{(a_{j_1},b_{j_1}),(a_{j_1^*},b_{j_1^*}),(c_{j_2},d_{j_2}),(c_{j_2^*},d_{j_2^*})\}$ is a perfect matching in $T(k_1,k_2)-F$. (3) Assume that there are two fault edges $(w_{j_1},w_{j_1^*})$ and $(w_{j_2},w_{j_2^*})$ in M_{w_0}. Let $a_0\in V(C_{k_1}^0)$ be the neighbour of w_0 such that a_0 and x_0 are distinct. Let $b_0\in V(C_{k_1}^0)$ be the neighbour of a_0 such that w_0 and b_0 are distinct. Then $M^*\cup\{(w_{j_1},a_{j_1}),(w_{j_1^*},a_{j_1^*}),(b_{j_1},b_{j_1^*}),(w_{j_2},a_{j_2}),(w_{j_2^*},a_{j_2^*}),(b_{j_2},b_{j_2^*})\}\backslash\{(w_{j_1},w_{j_1^*}),(a_{j_1},b_{j_1}),(a_{j_1^*},b_{j_1^*}),(w_{j_2},w_{j_2^*}),(a_{j_2},b_{j_2}),(a_{j_2^*},b_{j_2^*})\}$ is a perfect matching in $T(k_1,k_2)-F$. (4) Assume that there is exactly one fault edge $(w_{j_1},w_{j_1^*})$ in M_{w_0}. Let $a_0\in V(C_{k_1}^0)$ be the neighbour of w_0 such that a_0 and x_0 are distinct. If the other fault edge is incident to a_{j_1} or $a_{j_1^*}$ (say a_{j_1}), then $(w_{j_1^*},a_{j_1^*})$ is not faulty. Let $z_0\in V(C_{k_1}^0)$ be the neighbour of x_0 such that z_0 and w_0 are distinct. Let M_s be a perfect matching in $C_{k_2}(s)-s$ for each $s\in\{z_{i^*},w_{j_1^*},a_{j_1^*}\}$. Let M_0^* be a perfect matching in $C_{k_1}^0-\{x_0,w_0,a_0,z_0\}$ and let M_m^* be the corresponding matching to M_0^* for $m\in\{1,\dots,k_2-1\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i^*)\cup(\bigcup_{s\in\{z_{i^*},w_{j_1^*},a_{j_1^*}\}} M_s)\cup M_{x_0}\cup\{(x_{i^*},z_{i^*}),(w_{j_1^*},a_{j_1^*})\}$ is a perfect matching in $T(k_1,k_2)-F$. Next, we consider that the other fault edge is not incident to a_{j_1} or $a_{j_1^*}$. Let $b_0\in V(C_{k_1}^0)$ be the neighbour of a_0 such that w_0 and b_0 are distinct. Note that there is at most one fault edge (c_{j_2},d_{j_2}) in $\bigcup_{i=0}^{k_2-1} M_i$. We have $\{c_{j_2},d_{j_2}\}\cap\{a_{j_1},a_{j_1^*},b_{j_1},b_{j_1^*}\}=\emptyset$. We consider the following two subcases: (1) $\{c_{j_2},d_{j_2}\}\cap\{a_{j_1},a_{j_1^*},b_{j_1},b_{j_1^*}\}=\emptyset$ and there exists $j_2^*\in\{j_2+1,j_2-1\}$ such that $j_2^*\notin\{j_1,j_1^*\}$. Then $M^*\cup\{(w_{j_1},a_{j_1}),(w_{j_1^*},a_{j_1^*}),(b_{j_1},b_{j_1^*}),(c_{j_2},c_{j_2^*}),(d_{j_2},d_{j_2^*})\}\backslash\{(w_{j_1},w_{j_1^*}),(a_{j_1},b_{j_1}),(a_{j_1^*},b_{j_1^*}),(c_{j_2},d_{j_2}),(c_{j_2^*},d_{j_2^*})\}$ is a perfect matching in $T(k_1,k_2)-F$. (2) $\{c_{j_2},d_{j_2}\}\cap\{a_{j_1},a_{j_1^*},b_{j_1},b_{j_1^*}\}=\emptyset$ and there does not exist $j_2^*\in\{j_2+1,j_2-1\}$ such that $j_2^*\notin\{j_1,j_1^*\}$. It is easy to see that $k_2=3$. Let $z_0\in V(C_{k_1}^0)$ be the neighbour of x_0 such that z_0 and w_0 are distinct. Let $M_{z_{i^*}}$ be a perfect matching in $C_{k_2}(z_{i^*})-z_{i^*}$. Let M_0' be a perfect matching in $C_{k_1}^0-\{x_0,z_0\}$ and let M_m' be the corresponding matching to M_0' for $m\in\{1,\dots,k_2-1\}$. Then $(c_{j_2},d_{j_2})\notin\bigcup_{i=0}^{k_2-1} M_i'$. So $(\bigcup_{i=0}^{k_2-1} M_i')\cup M_{x_0}\cup M_{z_{i^*}}\cup\{(x_{i^*},z_{i^*})\}$ is a perfect matching in $T(k_1,k_2)-F$.

Case 3.2.2. $y_0\neq x_0$.

Let $N_{C_{k_1}^0}(x_0)=\{a_0,b_0\}$ and $N_{C_{k_1}^0}(y_0)=\{c_0,d_0\}$ such that b_0 is disconnected from c_0 in $C_{k_1}^0-\{x_0,y_0\}$. Let P_{j_1} be the path from x_i to c_i in $C_{k_1}^i-y_i$ and let P_{j_2} be the path from d_i to b_i in $C_{k_1}^i-y_i$.

Assume that $C_{k_1}^0-\{x_0,y_0\}$ can be partitioned into a set of paths of length one. If i is even, then let $P^*=(C_{k_1}^0-x_0)\cup(\bigcup_{j=1}^{i-1}(C_{k_1}^j-(x_j,b_j)))\cup P_{j_1}\cup\{(b_0,b_1),(x_1,x_2),\dots,(x_{i-1},x_i)\}$ (see Fig. 6(a)). If i is odd, then let $P^*=(C_{k_1}^0-x_0)\cup(\bigcup_{j=i+1}^{k_2-1}(C_{k_1}^j-(x_j,b_j)))\cup P_{j_1}\cup\{(b_0,b_{k_2-1}),(x_{k_2-1},x_{k_2-2}),\dots,(x_{i+1},x_i)\}$.

Assume that $C_{k_1}^0-\{x_0,y_0\}$ can be partitioned into a set of paths of length one plus two single vertices. If i is even, then let $P^*=(C_{k_1}^0-x_0)\cup(\bigcup_{j=i+1}^{k_2-1}(C_{k_1}^j-(x_j,b_j)))\cup P_{j_2}\cup\{(b_0,b_{k_2-1}),(x_{k_2-1},x_{k_2-2}),\dots,(b_{i+1},b_i)\}$ (see Fig. 6(b)). If i is odd, then let $P^*=(C_{k_1}^0-x_0)\cup(\bigcup_{j=1}^{i-1}(C_{k_1}^j-(x_j,b_j)))\cup P_{j_2}\cup\{(b_0,b_1),(x_1,x_2),\dots,(b_{i-1},b_i)\}$.

Note that P^* is an odd path and $C_{k_1}^i-(V(P^*)\cup\{y_i\})$ is an odd path. Thus there exist perfect matchings M^* and M_{y_i} in P^* and $C_{k_1}^i-(V(P^*)\cup\{y_i\})$, respectively. We consider the following three subcases.

Case 3.2.2.1. There is no fault edge in M^* or there is exactly one fault edge in M_{y_i} and M^*, respectively.

Suppose that there is no fault edge in M_{y_i}. If there exists some $j\in\{1,\dots,k_2-1\}\backslash\{i\}$ such that $E(C_{k_2}^j)\cap M^*=\emptyset$ and $C_{k_2}^j$ contains two fault edges, then any perfect matching in $C_{k_2}^j$ contains at most one fault edge. For $l\in\{j-1,j,j+1\}$, let M_l be the perfect matching in $C_{k_1}^l$ such that $M_0\cap M^*\neq\emptyset$, where M_0 is the corresponding matching to M_l. Let (u_j,v_j) be the fault edge in M_j. Then there exists $j^*\in\{j+1,j-1\}$ such that u_{j^*} and v_{j^*} are not fault vertices. When $M_{j^*}\cap M^*=\emptyset$, completing $M^*\cup M_{y_i}\cup M_j\cup M_{j^*}\cup\{(u_j,u_{j^*}),(v_j,v_{j^*})\}\backslash\{(u_j,v_j),(u_{j^*},v_{j^*})\}$ gives a perfect matching of $T(k_1,k_2)-F$. When

$M_{j^*} \cap M^* \neq \emptyset$, completing $M^* \cup M_{y_i} \cup M_j \cup \{(u_j, u_{j^*}), (v_j, v_{j^*})\} \backslash \{(u_j, v_j), (u_{j^*}, v_{j^*})\}$ gives a perfect matching of $T(k_1, k_2) - F$. Otherwise, completing $M^* \cup M_{y_i}$ gives a perfect matching of $T(k_1, k_2) - F$.

Suppose that there are two fault edges (u_i, v_i) and (w_i, z_i) in M_{y_i}. If there exists $i^* \in \{i+1, i-1\}$ such that $C_{k_1}^{i^*} \cap M^* = \emptyset$, then $C_{k_1}^{i^*} - \{u_{i^*}, v_{i^*}, w_{i^*}, z_{i^*}\}$ has a perfect matching M_{i^*} and completing $M^* \cup M_{y_i} \cup M_{i^*} \cup \{(u_i, u_{i^*}), (v_i, v_{i^*}), (w_i, w_{i^*}), (z_i, z_{i^*})\} \backslash \{(u_i, v_i), (w_i, z_i)\}$ gives a perfect matching of $T(k_1, k_2) - F$. Otherwise, $C_{k_1}^{i+1} \cap M^* \neq \emptyset$ and $C_{k_1}^{i-1} \cap M^* \neq \emptyset$. Choose $i^* \in \{i+1, i-1\}$ such that $i^* \neq 0$. Let $C^* = (C_{k_1}^{i} - y_i) \cup \{(c_i, c_{i^*}), (c_{i^*}, y_{i^*}), (y_{i^*}, d_{i^*}), (d_{i^*}, d_i)\}$. Then there exists a perfect matching M_{C^*} in C^* such that $\{(u_i, v_i), (w_i, z_i)\} \nsubseteq M_{C^*}$ and $C_{k_1}^{i^*} - \{c_{i^*}, y_{i^*}, d_{i^*}, b_{i^*}\}$ can be partitioned into a set M_{i^*} of paths of length one. Thus, $(M^* \backslash (E(C_{k_1}^{i^*}) \cup E(C_{k_1}^{i}))) \cup M_{C^*} \cup M_{i^*} \backslash \{(x_{i^*}, x_i)\}$ gives a perfect matching of $T(k_1, k_2) - F$.

Suppose that there is exactly one fault edge (u_i, v_i) in M_{y_i}. If $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one, then $u_0 \neq x_0$ and $v_0 \neq x_0$. For any $z \in \{x_0, y_i\} \cup V(C_{k_2}(u_i)) \cup V(C_{k_2}(v_i))$, let M_z be the perfect matching in $C_{k_2}(z) - z$. Note that there is at most one fault edge in $T(k_1, k_2) - \{x_0, y_i, (u_i, v_i)\}$. So there exists $i^* \in \{0, 1, \dots, k_2 - 1\}$ such that $M_{u_{i^*}} \cup M_{v_{i^*}} \cup \{(u_{i^*}, v_{i^*})\}$ contains no fault edges. Let M_0 be a perfect matching in $C_{k_1}^0 - \{x_0, y_0, u_0, v_0\}$ and let M_j be the corresponding matching to M_0 for $j \in \{1, \dots, k_2 - 1\}$. Let $M' = (\bigcup_{j=0}^{k_2-1} M_j) \cup M_{x_0} \cup M_{y_i} \cup M_{u_{i^*}} \cup M_{v_{i^*}} \cup \{(u_{i^*}, v_{i^*})\}$. When $(\bigcup_{j=0}^{k_2-1} M_j) \cup M_{x_0} \cup M_{y_i}$ contains no fault edges, M' is a perfect matching in $T(k_1, k_2) - F$. When the other fault edge is in $\bigcup_{j=0}^{k_2-1} M_j$, without loss of generality, say (w_l, z_l) is the fault edge, $M' \cup \{(w_l, w_{l+1}), (z_l, z_{l+1})\} \backslash \{(w_l, z_l), (w_{l+1}, z_{l+1})\}$ is a perfect matching in $T(k_1, k_2) - F$. When the other fault edge is in $M_{x_0} \cup M_{y_i}$, without loss of generality, say (x_l, x_{l+1}) is the fault edge and $a_i \notin \{u_i, v_i\}$. Let $w_l \in N_{C_{k_1}^l}(a_l)$ such that $w_l \neq x_l$. Then $M' \cup \{(x_l, a_l), (x_{l+1}, a_{l+1}), (w_l, w_{l+1})\} \backslash \{(a_l, w_l), (a_{l+1}, w_{l+1}), (x_l, x_{l+1})\}$ is a perfect matching in $T(k_1, k_2) - F$.

Otherwise, when there is no fault edge in $C_{k_1}^0 - x_0$, similarly to the proof of Case 3.1.1.2, we can obtain a perfect matching M'' in $T(k_1, k_2) - F$; when the other fault edge is (w_0, z_0), either M'' or $M'' \cup \{(w_0, w_l), (z_0, z_l)\} \backslash \{(w_0, z_0), (w_l, z_l)\}$ for some $l \in \{1, k_2 - 1\}$ is a perfect matching in $T(k_1, k_2) - F$.

Case 3.2.2.2. There is exactly one fault edge in M^* and there is no fault edge in M_{y_i}.

Consider when F is not a trivial strong matching preclusion set. Assume that the fault edge in M^* is not a cross edge. Let (u_j, v_j) be the fault edge in M^*. Suppose that $j \in \{i+1, i-1\}$, without loss of generality, say $j = i - 1$. If $y_j = v_j$ and (v_j, v_{j-1}) is the other fault edge, then we consider the following two subcases: (1) when $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one, without loss of generality, say $u_j = d_j$ and $E(C_{k_1}^{j-1}) \cap M^* \neq \emptyset$. Suppose that $c_j \neq x_j$. Let $w_j \in V(C_{k_1}^j)$ be the neighbour of c_j such that v_j and w_j are distinct. Then completing $M^* \cup M_{y_i} \cup \{(u_j, u_{j-1}), (v_j, c_j), (v_{j-1}, c_{j-1}), (w_j, w_{j-1})\} \backslash \{(w_{j-1}, c_{j-1}), (w_j, c_j), (u_{j-1}, v_{j-1}), (u_j, v_j)\}$ gives a perfect matching of $T(k_1, k_2) - F$. Suppose that $c_j = x_j$. $C_{k_1}^0 - \{x_0, y_0, b_0, d_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_m be the corresponding matching to M_0 for $m \in \{1, \dots, k_2 - 1\}$. Let M_{y_0} be a perfect matching in $C_{k_2}(y_0) - \{y_{j-1}, y_j, y_i\}$. Let M_{x_0} be a perfect matching in $C_{k_2}(x_0) - \{x_0, x_j, x_i\}$. Let M_z be a perfect matching in $C_{k_2}(z) - z$ for each $z \in \{d_{j-1}, b_i\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{x_0, y_0, d_{j-1}, b_i\}} M_z) \cup \{(y_j, x_j), (x_i, b_i), (y_{j-1}, d_{j-1})\}$ gives a perfect matching of $T(k_1, k_2) - F$. (2) When $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one plus two single vertices, the even cycle $C = (c_{i-1}, c_i, c_{i+1}, y_{i+1}, d_{i+1}, d_i, d_{i-1}, y_{i-1}, c_{i-1})$ contains one fault edge. So there exists a perfect matching M_C in C. Let M_0 be the perfect matching in $C_{k_1}^0 - \{x_0, c_0, y_0, d_0\}$ and let M_j be the corresponding matching to M_0 for $j \in \{1, \dots, k_2 - 1\}$. Let M_{x_0} be the perfect matching in $C_{k_2}(x_0) - x_0$. Let M_{z_0} be the perfect matching in $C_{k_2}(z_0) - \{z_{i-1}, z_i, z_{i+1}\}$ for each $z_0 \in \{c_0, d_0, y_0\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup (\bigcup_{z \in \{x_0, c_0, d_0, y_0\}} M_z) \cup M_C$ is a perfect matching in $T(k_1, k_2) - F$.

Otherwise, there exists $j^* \in \{i+1, i-1\}$ such that (u_{j^*}, u_j) and (v_{j^*}, v_j) are not fault edges. Let M_{j^*} be the perfect matching in $C_{k_1}^{j^*}$ such that $M_0 \cap M^* \neq \emptyset$, where M_0 is the corresponding matching to M_{j^*}. When $M_{j^*} \cap M^* = \emptyset$, completing $M^* \cup M_{y_i} \cup M_{j^*} \cup \{(u_j, u_{j^*}), (v_j, v_{j^*})\} \backslash \{(u_j, v_j), (u_{j^*}, v_{j^*})\}$ gives a perfect matching of $T(k_1, k_2) - F$. When $M_{j^*} \cap M^* \neq \emptyset$, completing $M^* \cup M_{y_i} \cup \{(u_j, u_{j^*}), (v_j, v_{j^*})\} \backslash \{(u_j, v_j), (u_{j^*}, v_{j^*})\}$ gives a perfect matching of $T(k_1, k_2) - F$.

Assume that the fault edge in M^* is a cross edge. Suppose that $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one. Let $u_0 \in V(C_{k_1}^0)$ be the neighbour of b_0 such that x_0 and u_0 are distinct. Let $v_0 \in V(C_{k_1}^0)$ be the neighbour of u_0 such that v_0 and b_0 are distinct. If $x_0 \notin \{c_0, d_0\}$, then we consider the following two subcases. Without loss of generality, say (b_j, b_{j+1}) is the fault cross edge. (1) When (b_j, u_j), (b_{j+1}, u_{j+1}) and (v_j, v_{j+1}) are not fault edges, completing $M^* \cup M_{y_i} \cup \{(b_j, u_j), (b_{j+1}, u_{j+1}), (v_j, v_{j+1})\} \backslash \{(v_j, u_j), (v_{j+1}, u_{j+1}), (b_j, b_{j+1})\}$ gives a perfect matching of $T(k_1, k_2) - F$. (2) When one edge in $\{(b_j, u_j), (b_{j+1}, u_{j+1}), (v_j, v_{j+1})\}$ is the other fault edge, $C_1 = (a_{k_2-1}, a_0, a_1, x_1, b_1, b_0, b_{k_2-1}, x_{k_2-1}, a_{k_2-1})$ contains at most one fault edge and $C_2 = (c_{i-1}, c_i, c_{i+1}, y_{i+1}, d_{i+1}, d_i, d_{i-1}, y_{i-1}, c_{i-1})$ contains at most one fault edge. So there exist perfect matchings M_{C_1} and M_{C_2} in C_1 and C_2, respectively. $C_{k_1}^0 - \{x_0, a_0, b_0, c_0, y_0, d_0\}$ can be partitioned into a set M_0 of paths of length one. Let M_j be the corresponding matching to M_0 for $j \in \{1, \dots, k_2 - 1\}$. Let M_{t_0} be the perfect matching in $C_{k_2}(t_0) - \{t_{k_2-1}, t_0, t_1\}$ for each $t_0 \in \{a_0, b_0, x_0\}$. Let M_{z_0} be the perfect matching in $C_{k_2}(z_0) - \{z_{i-1}, z_i, z_{i+1}\}$ for each $z_0 \in \{c_0, d_0, y_0\}$. Then $(\bigcup_{j=0}^{k_2-1} M_j) \cup (\bigcup_{z \in \{a_0, b_0, x_0, c_0, d_0, y_0\}} M_z) \cup M_{C_1} \cup M_{C_2}$ is a perfect matching in $T(k_1, k_2) - F$.

If $x_0 \in \{c_0, d_0\}$ (say $x_0 = c_0$), then we consider the following four cases: (1) assume that (b_j, b_{j+1}) is the fault cross edge and there is no fault edge in $\{(b_j, u_j), (b_{j+1}, u_{j+1}), (v_j, v_{j+1})\}$. By the similar way of (1) in the above paragraph, we can obtain a perfect matching in $T(k_1, k_2) - F$. (2) Assume that (b_j, b_{j+1}) is the fault cross edge and the other fault edge is in $\{(b_j, u_j), (b_{j+1}, u_{j+1}), (v_j, v_{j+1})\}$. Let $C = (C_{k_2}(b_0) - (b_j, b_{j+1})) \cup (C_{k_2}(u_0) - (u_j, u_{j+1})) \cup \{(u_j, b_j), (u_{j+1}, b_{j+1})\}$. Then C contains one fault edge and there is a perfect matching M_C in C. Let M_0 be the perfect matching in $C_{k_1}^0 - \{x_0, b_0, y_0, u_0\}$ and let M_j be the corresponding matching to M_0 for $j \in \{1, \ldots, k_2 - 1\}$. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{x_0, y_i\}$. Then $(\bigcup_{i=0}^{k_2-1} M_i) \cup M_{x_0} \cup M_{y_i} \cup M_C$ is a perfect matching in $T(k_1, k_2) - F$. (3) Assume that (x_j, x_{j+1}) is the fault cross edge and $i \notin \{j, j+1\}$. Let M_0 be the perfect matching in $C_{k_1}^0 - \{x_0, y_0\}$ and let M_j be the corresponding matching to M_0 for $j \in \{1, \ldots, k_2 - 1\}$. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{x_0, y_i\}$. Then $(\bigcup_{j=0}^{k_2-1} M_j) \cup M_{x_0} \cup M_{y_i} \cup \{(x_j, b_j), (x_{j+1}, b_{j+1}), (u_j, u_{j+1})\} \backslash \{(x_j, x_{j+1}), (u_j, b_j), (u_{j+1}, b_{j+1})\}$ is a perfect matching in $T(k_1, k_2) - F$ or we can obtain a perfect matching in $T(k_1, k_2) - F$ by the similar way of (1) in the above paragraph. (4) Assume that (x_j, x_{j+1}) is the fault cross edge and $i \in \{j, j+1\}$ (say $i = j$). Note that there is one fault edge in $T(k_1, k_2) - \{x_0, y_0, (x_j, x_{j+1})\}$. When (x_j, b_j) and (x_{j+1}, y_{j+1}) are not fault edges, $C = (C_{k_1}^j - \{x_j, y_j, b_j\}) \cup (C_{k_1}^{j+1} - \{x_{j+1}, y_{j+1}, b_{j+1}\}) \cup \{(u_j, u_{j+1}), (d_j, d_{j+1})\}$ contains at most one fault edge. So C has a perfect matching M_C. Thus, completing $M^* \cup M_C \cup \{(x_j, b_j), (x_{j+1}, y_{j+1})\} \backslash (E(C_{k_1}^{j+1}) \cup \{(x_j, x_{j+1})\})$ gives a perfect matching of $T(k_1, k_2) - F$. Let M_0 be the perfect matching in $C_{k_1}^0 - \{x_0, b_0, y_0, u_0\}$ and let M_j be the corresponding matching to M_0 for $j \in \{1, \ldots, k_2 - 1\}$. Let M_z be the perfect matching in $C_{k_2}(z) - z$ for each $z \in \{u_0, y_i\}$. When (x_{j+1}, y_{j+1}) is the other fault edge, $(\bigcup_{j=0}^{k_2-1} M_j) \cup (\bigcup_{j=1}^{k_2-1} \{(x_j, b_j)\}) \cup M_{u_0} \cup M_{y_i} \cup \{(b_0, u_0)\}$ is a perfect matching in $T(k_1, k_2) - F$. When (x_j, b_j) is the other fault edge, we have $j - 1 \neq 0$. Thus, $(\bigcup_{l=0}^{k_2-1} M_l) \cup (\bigcup_{l \in \{1, \ldots, k_2-1\} \backslash \{j-1, j\}} \{(x_l, b_l)\}) \cup M_{u_0} \cup M_{y_i} \cup \{(b_0, u_0), (x_{j-1}, x_j), (b_{j-1}, b_j)\}$ is a perfect matching in $T(k_1, k_2) - F$.

Suppose that $C_{k_1}^0 - \{x_0, y_0\}$ can be partitioned into a set of paths of length one plus two single vertices. Assume that either $C_{k_2}(x_0)$ contains no fault edges or $C_{k_2}(y_0)$ contains no fault edges. If there is no fault edge in $C_{k_1}^0 - x_0$, then, similarly to the proof of Case 3.1.1.2, we can obtain a perfect matching M in $T(k_1, k_2) - F$; if the other fault edge is (w_0, z_0), then either M or $M \cup \{(w_0, w_l), (z_0, z_l)\} \backslash \{(w_0, z_0), (w_l, z_l)\}$ for some $l \in \{1, k_2 - 1\}$ is a perfect matching in $T(k_1, k_2) - F$. Assume that $C_{k_2}(x_0)$ contains one fault edge (x_j, x_{j+1}) and $C_{k_2}(y_0)$ contains at most one fault edge. Let $g_0 \in V(C_{k_1}^0)$ be the neighbour of a_0 such that x_0 and g_0 are distinct. Then completing $M^* \cup M_{y_i} \cup \{(x_j, a_j), (x_{j+1}, a_{j+1}), (g_j, g_{j+1})\} \backslash \{(a_j, g_j), (a_{j+1}, g_{j+1}), (x_j, x_{j+1})\}$ gives a perfect matching of $T(k_1, k_2) - F$.

Case 3.2.2.3. There are two fault edges in M^*.

Suppose that the two fault edges are cross edges. Then there exists a fault cross edge (b_j, b_{j+1}) such that $\{c_i, d_i\} \cap \{b_j, b_{j+1}\} = \emptyset$ or there exists a fault cross edge (x_j, x_{j+1}) such that $\{c_i, d_i\} \cap \{x_j, x_{j+1}\} = \emptyset$. Without loss of generality, say there exists a fault cross edge (b_j, b_{j+1}) such that $\{c_i, d_i\} \cap \{b_j, b_{j+1}\} = \emptyset$. Let $u_0 \in V(C_{k_1}^0)$ be the neighbour of b_0 such that x_0 and u_0 are distinct. Let $v_0 \in V(C_{k_1}^0)$ be the neighbour of u_0 such that v_0 and b_0 are distinct. Then completing $M^* \cup M_{y_i} \cup \{(b_j, u_j), (b_{j+1}, u_{j+1}), (v_j, v_{j+1})\} \backslash \{(v_j, u_j), (v_{j+1}, u_{j+1}), (b_j, b_{j+1})\}$ gives a perfect matching M' of $T(k_1, k_2) - \{x_0, y_i, (b_j, b_{j+1})\}$. If the other fault edge satisfies the above condition, then, by repeating the above operation, we can obtain a perfect matching in $T(k_1, k_2) - F$. Otherwise, without loss of generality, the other fault edge (b_{j_1}, b_{j_1+1}) satisfies $d_i \in \{b_{j_1}, b_{j_1+1}\}$. There exists $i^* \in \{i+1, i-1\}$ such that $E(C_{k_1}^{i^*}) \cap M^* \neq \emptyset$. Now, $C = (C_{k_1}^i - y_i) \cup \{(c_i, c_{i^*}), (c_{i^*}, y_{i^*}), (y_{i^*}, d_{i^*}), (d_{i^*}, d_i)\}$ is an even cycle containing one fault edge. So C has a perfect matching M_C. Thus, $M' \cup M_C \backslash (M_{y_i} \cup \{(d_i, d_{i^*}), (c_{i^*}, y_{i^*})\})$ is a perfect matching M'' in $T(k_1, k_2) - F$.

Suppose that one of the two fault edges is not a cross edge, without loss of generality, say (w_l, z_l) is a fault edge. Then there exists $l^* \in \{l+1, l-1\}$ such that w_{l^*} and z_{l^*} are not fault vertices. Let M_{l^*} be the perfect matching in $C_{k_1}^{l^*}$ such that $M_0 \cap M^* \neq \emptyset$, where M_0 is the corresponding matching to M_{l^*}. When $M_{l^*} \cap M^* = \emptyset$, completing $M^* \cup M_{y_i} \cup M_{l^*} \cup \{(w_l, w_{l^*}), (z_l, z_{l^*})\} \backslash \{(w_l, z_l), (w_{l^*}, z_{l^*})\}$ gives a perfect matching M''' of $T(k_1, k_2) - \{x_0, y_i, (w_l, z_l)\}$. When $M_{l^*} \cap M^* \neq \emptyset$, completing $M^* \cup M_{y_i} \cup \{(w_l, w_{l^*}), (z_l, z_{l^*})\} \backslash \{(w_l, z_l), (w_{l^*}, z_{l^*})\}$ gives a perfect matching M''' of $T(k_1, k_2) - \{x_0, y_i, (w_l, z_l)\}$. If the other fault edge is not a cross edge and is not in $\{(w_{l+1}, z_{l+1}), (w_{l-1}, z_{l-1})\}$, then, by repeating the above operation, we can obtain a perfect matching in $T(k_1, k_2) - F$. If the other fault edge is (w_{l^*}, z_{l^*}), where $l^* \in \{l+1, l-1\}$, then M''' is a perfect matching in $T(k_1, k_2) - F$. If the other fault edge is a cross edge, then we can obtain a perfect matching in $T(k_1, k_2) - F$ by the similar way in the above paragraph.

Case 4. $|F_v| = 1$.

In this case, $|F_e| \leqslant 3$. By Lemma 3.2, $mp(T(k_1, k_2)) = 4$. So there exists a perfect matching M in $T(k_1, k_2) - F_e$. Note that M saturates all the vertices in $V(T(k_1, k_2))$ and one fault vertex can damage exactly one edge in M. Let $e^* \in M$ be the edge such that e^* is incident to the fault vertex. Then $M \backslash \{e^*\}$ gives an almost perfect matching of $T(k_1, k_2) - F$.

Case 5. $|F_v| = 0$.

In this case, $F = F_e$ and $|F| = |F_e| \leqslant 4$. By Lemma 3.2, $mp(T(k_1, k_2)) = 4$ and each of its minimum MP sets is trivial. So if F_e is not a trivial strong matching preclusion set, then $T(k_1, k_2) - F$ has a perfect matching. Thus, either $T(k_1, k_2) - F$ is matchable or F is a trivial strong matching preclusion set. $\square$

Lemma 3.3. *(See [10].) Let $k_1 \geqslant 3$ and let $k_2 \geqslant 3$ be odd. Then $T(k_1, k_2) - F$ has a Hamiltonian cycle for any fault set F with $|F| \leqslant 2$.*

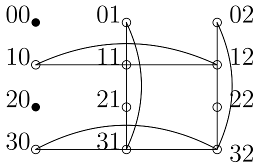

Fig. 7. The faulty $T(4,3)$ in Example 3.1.

Let $k \geqslant 3$ be an odd integer. Consider a fault set F in $T(4,k)$ with $|F| = 3$. By Lemma 3.3, $T(4,k) - F$ is matchable, which means $smp(T(4,k)) > 3$. By Proposition 1.3, $smp(T(4,k)) \leqslant \delta(T(4,k)) = 4$. So $smp(T(4,k)) = 4$, i.e., $T(4,k)$ is maximally strong matched. However, $T(4,k)$ is not super strong matched. See the following example (see Fig. 7).

Example 3.1. Let $T(4,3) = (0,1,2,3,0) \times (0,1,2,0)$ be a 2-dimensional torus. Let $F_v = \{00, 20\}$ and $F_e = \{(01,02),(21,22)\}$. It is easy to see that there is no perfect matching in $T(4,3) - (F_v \cup F_e)$ and $F_v \cup F_e$ is not a trivial strong matching preclusion set.

4. Conclusion

In this paper, we studied the strong matching preclusion for torus networks. We establish the strong matching preclusion number and all possible minimum strong matching preclusion sets for bipartite torus networks and 2-dimensional nonbipartite torus networks. The results can be used in robustness analysis for torus networks with respect to the property of having a perfect matching or an almost perfect matching. Our further work is to investigate the problem of strong matching preclusion for n-dimensional nonbipartite torus networks, where $n \geqslant 3$.

Acknowledgements

The authors would like to express their deepest gratitude to the anonymous referees for the constructive suggestions and comments that improve the quality of this paper.

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