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Consistency measure, inclusion degree and fuzzy measure in decision tables

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Abstract

Classical consistency degree has some limitations for measuring the consistency of a decision table, in which the lower approximation of a target decision is only taken into consideration. In this paper, we focus on how to measure the consistencies of a target concept and a decision table and the fuzziness of a rough set and a rough decision in rough set theory. For three types of decision tables (complete, incomplete and maximal consistent blocks), the membership functions of an object are defined through using the equivalence class, tolerance class and maximal consistent blocks including itself, respectively. Based on these membership functions, we introduce consistency measures to assess the consistencies of a target set and a decision table, and define fuzziness measures to compute the fuzziness of a rough set and a rough decision in these three types of decision tables. In addition, the relationships among the consistency, inclusion degree and fuzzy measure are established as well. These results will be helpful for understanding the essence of the uncertainty in decision tables and can be applied for rule extraction and rough classification in practical decision issues.

Keywords: Rough set theory; Decision table; Consistency measure; Fuzziness measure; Inclusion degree

1. Introduction

Rough set theory proposed by Pawlak in [21] is a relatively new soft computing tool for the analysis of a vague description of an object, and has become a popular mathematical framework for pattern recognition, image processing, feature selection, neuro computing, conflict analysis, decision support, data mining and knowledge discovery from large data sets [22,23,26]. Rough-set-based data analysis starts from a data table, called information systems. The information systems contain data about objects of interest, characterized by a finite set of attributes. It is often interesting to discover some dependency relationships (patterns). An information system where condition attributes and decision attributes are distinguished is called a decision table. From a decision table one can induce some patterns in form of "*if* ..., *then* ..." decision rules. More exactly, the decision rules say that if some condition attributes have given values, then some decision attributes have other given values.

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In many practical issues, it may happen that some of the attribute values for an object are missing in an information system. For example, in medical information systems there may exist a group of patients for which it is impossible to perform all the required tests. These missing values can be represented by the set of all possible values for the attribute or equivalence by the domain of the attribute. To indicate such a situation, a distinguished value, the so-called null value, is usually assigned to those attributes. According to whether or not there are missing data (null values), information systems can be classified into two categories: complete and incomplete. By an incomplete information system we mean a system with missing data (null values). In this paper, we will only deal with the case of unknown values in which a null value may be some value in the domain of the corresponding attribute [7,8,10,24]. For the case that a null value means an inapplicable value, it can be handled by adding to the attribute domains a special symbol for inapplicable values. For an incomplete information system, if condition attributes and decision attributes are distinguished, then it is called an incomplete decision table.

A set of decision rules can be generated from a decision table for classification and prediction by adopting any kind of reduct technique in rough set theory [6,30,41]. In a broad sense, a reduct can keep the consistency of an information system or a decision table. In the past 20 years, many kinds of reduct techniques for information systems and decision tables have been proposed in rough set theory [1,9,10,15–18,20–22,29,31–36,43,42]. β-Reduct proposed by Ziarko provides a kind of attribute reduction method in the variable precision rough set model [43]. α-Reduct and α-relative reduct, which allow the occurrence of additional inconsistency, are proposed in [20] for information systems and decision tables, respectively. An attribute reduction method preserving the class membership distribution for all objects in information systems was proposed by Slezak in [31,32]. Five kinds of attribute reducts and the relationships among them in inconsistent systems are investigated by Kryszkiewicz [9], Li [11] and Mi [18]. By eliminating the rigorous conditions required by distribution reduct, maximum distribution reduct is introduced by Mi in [19]. Unlike possible reduct [19], maximum distribution reduct can derive decision rules that are compatible with the original systems.

The predictive performance on a set of unseen examples is often the key aspect to determine which of the rule extraction methods should be preferred for a particular application. There are several important criterions, such as scalability, comprehensibility and consistency that influence the suitability of an algorithm for a given problem. The existing definitions of consistency all try to find out whether a repeated application of the rule learning algorithm on a data set will provide similar results. The various definitions vary on what exactly they consider to be the result: the accuracy, the predictions, similarity between two rules or the rule set itself [5]. In this paper, we focus on the aspect of consistency in the context of decision tables.

Because the notions of approximation accuracy of decision classes and consistency degree [21,23] are defined for a decision table, in some sense, they could be regarded as measures for evaluating the decision performance of all decision rules generated from the decision table [25,27]. Nevertheless, the approximation accuracy and the consistency degree have some limitations. For instance, the certainty and consistency of a decision-rule set could not be well depicted by the approximation accuracy and the consistency degree when their values achieve zero. As we know, the fact that approximation accuracy/consistency degree is equal to zero only implies that there is no decision rule with the certainty of one in the decision table. Hence, the approximation accuracy and the consistency degree of a decision table cannot give elaborate depictions of the certainty and consistency for a rule set from the decision table. Therefore, we introduced three new measures (α, β and γ) to assess the entire decision performance of a decision-rule set extracted from a complete decision table [25,27]. So far, however, how to assess the consistency of an incomplete decision table and that of a decision table in the context of maximal consistent blocks have not been reported.

Uncertainty is one of the main characteristics of a complex system. Since fuzzy set theory in which a crisp set is expanded to a fuzzy set was proposed in 1965, more and more research on uncertainties have appeared [3]. In recent years, the concept of inclusion degree was proposed in which uncertain relations of objects have been studied in detail [38,39]. The concept of inclusion degree has also been introduced into rough set theory and several important relationships between inclusion degree and measures of rough set data analysis have been established [37].

The concept of inclusion degree is derived from the including measure among sets. If $D(B/A)$ denotes the degree for set A included in set B, then the following properties hold:

(1) $0 \leqslant D(B/A) \leqslant 1$.
(2) If $A \subseteq B$ then $D(B/A) = 1$.
(3) When $B_1 \subseteq B_2$ then $D(B_1/A) \leqslant D(B_2/A)$ holds.

However, quite a few techniques for uncertainty reasoning employed in intelligent systems such as the probability reasoning method and the MYCIN uncertainty factor are unable to satisfy property (3). Qiu et al. modified condition (3) to fit the method better with the requirements of uncertainty reasoning [28]. The subsets $\mathbf{P}(U)$ on the domain U and the including relation $\subseteq$ form a poset $(\mathbf{P}(U), \subseteq)$ [28]. The inclusion degree on a poset is defined. The inclusion degree not only reflects including relations between sets but also shows the comparisons between different objects as well as the comparisons between numbers and vectors [40]. Qiu defined inclusion degrees on interval numbers, divisions, vectors and set vectors, and illustrates their validity and widespread applications in the uncertainty analysis of intelligent systems [28].

This paper aims to find methods for measuring the consistency of a target concept and a decision table and computing the fuzziness of a rough set and a rough decision in three types of decision tables. The rest of this paper is organized as follows. Some preliminary concepts such as complete decision tables, incomplete decision tables and maximal consistent block technique are briefly reviewed in Section 2. In Section 3, the concept of a consistency in the context of a complete decision table is defined and the relationships between the inclusion degree, the fuzzy measure and this definition are established. In Section 4, in incomplete decision tables, we introduce a consistency measure to calculate the degree of the condition part included in the decision part and define a fuzziness measure to compute the fuzziness of a rough set and a rough decision. In Section 5, a consistency measure and a fuzziness measure in the context of maximal consistent block technique are introduced to an incomplete decision table and their several properties are obtained. Section 6 concludes this paper with some remarks and discussions.

2. Preliminaries

In this section, we review some basic concepts such as information systems, incomplete information systems and maximal consistent blocks.

An information system is a pair $S = (U, A)$, where,

(1) U is a non-empty finite set of objects;
(2) A is a non-empty finite set of attributes; and
(3) for every $a \in A$, there is a mapping $a: U \to V_a$, where V_a is called the value set of a.

Each subset of attributes $P \subseteq A$ determines a binary indistinguishable relation $IND(P)$ given by

$$IND(P) = \{(u, v) \in U \times U \mid \forall a \in P, a(u) = a(v)\}.$$

It can be shown that $IND(P)$ is an equivalence relation on the set U. For $P \subseteq A$, the relation $IND(P)$ constitutes a partition of U, which is denoted by $U/IND(P)$, or just U/P.

It may happen that some of the attribute values for an object are missing. For example, in medical information systems there may exist a group of patients for which it is impossible to perform all the required tests. These missing values can be represented by the set of all possible values for the attribute or equivalence by the domain of the attribute. To indicate such a situation, a distinguished value (the so-called null value) is usually assigned to those attributes. If V_a contains a null value for at least one attribute $a \in A$, then S is called an incomplete information system, otherwise it is complete [7,8,13,14]. From now on, we will denote the null value by $*$.

Let $S = (U, A)$ be an information system and $P \subseteq A$ an attribute set. We define a binary relation on U by

$$SIM(P) = \{(u, v) \in U \times U \mid \forall a \in P, a(u) = a(v) \text{ or } a(u) = * \text{ or } a(v) = *\}.$$

In fact, $SIM(P)$ is a tolerance relation on U. The concept of a tolerance relation has a wide variety of applications in classifications [7,8,13,14]. It can be easily shown that $SIM(P) = \bigcap_{a \in P} SIM(\{a\})$. Let $S_P(u)$ denote the set $\{v \in U | (u, v) \in SIM(P)\}$. Then, $S_P(u)$ is the maximal set of objects which are possibly indistinguishable by P with u. Let $U/SIM(P)$ denote the family sets $\{S_P(u) | u \in U\}$, which is the classification or the knowledge induced by P. A member $S_P(u)$ from $U/SIM(P)$ will be called a tolerance class or an information granule. It should be noticed that the tolerance classes in $U/SIM(P)$ do not constitute a partition of U in general. They constitute a cover of U, i.e., $S_P(u) \neq \emptyset$ for every $u \in U$, and $\bigcup_{u \in U} S_P(u) = U$.

An incomplete information system $S = (U, C \cup D)$ is called an incomplete decision table if condition attributes and decision attributes are distinguished, where C is the condition attribute set and D is the decision attribute set. Naturally,

Table 1
The incomplete decision table about car [8]

Car	Price	Mileage	Size	Max-Speed	d	∂_d
u_1	High	Low	Full	Low	Good	{good}
u_2	Low	*	Full	Low	Good	{good}
u_3	*	*	Compact	Low	Poor	{poor}
u_4	High	*	Full	High	Good	{good, excellent}
u_5	*	*	Full	High	Excellent	{good, excellent}
u_6	Low	High	Full	*	Good	{good, excellent}

a complete information system $S = (U, C \cup D)$ is called a complete decision table. This is illustrated in the following example.

Example 1. Consider the descriptions of several cars in Table 1 [8].

This is an incomplete decision table, where $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, $C = \{a_1, a_2, a_3, a_4\}$ with a_1—Price, a_2—Mileage, a_3—Size, a_4—Max-Speed, and $D = \{d\}$. By computing, it follows that

$$U/SIM(C) = \{S_C(u_1), S_C(u_2), S_C(u_3), S_C(u_4), S_C(u_5), S_C(u_6)\},$$

where $S_C(u_1) = \{u_1\}$, $S_C(u_2) = \{u_2, u_6\}$, $S_C(u_3) = \{u_3\}$, $S_C(u_4) = \{u_4, u_5\}$, $S_C(u_5) = \{u_4, u_5, u_6\}$, $S_C(u_6) = \{u_2, u_5, u_6\}$.

It is trivial to observe that the value of the generalized decision ∂_d for an object in an incomplete decision table is the superset of the object's value (see ∂_d in Table 1).

However, tolerance classes are not the minimal units for describing knowledge or information in incomplete information systems [4,10]. Let $S = (U, A)$ be an information system, $P \subseteq A$ an attribute set and $X \subseteq U$ a subset of objects. We say X is consistent with respect to P if $(u, v) \in SIM(P)$ for any $u, v \in X$. If there does not exist a subset $Y \subseteq U$ such that $X \subset Y$ and Y is consistent with respect to P, then X is called a maximal consistent block of P. Obviously, in a maximal consistent block, all objects are indiscernible with available information provided by a similarity relation [10]. Henceforth, we denote by MC_P the set of all maximal consistent blocks determined by $P \subseteq A$, and by $MC_P(u)$ the set of all maximal consistent blocks of P which includes some object $u \in U$, respectively. It is clear that $X \in MC_P$ if and only if $X = \bigcap_{u \in X} S_P(u)$ [10]. This is illustrated in Example 2. In fact, the set of all maximal consistent blocks MC_P will degenerate into the partition U/P induced by the attribute set P in a complete information system, i.e., $MC_P = U/P$.

Example 2. Compute all maximal consistent blocks of C in Table 1.

By computing, from Example 1, we have that

$$MC_C = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\},$$

where MC_C is the set of all maximal consistent blocks determined by C on U.

3. The consistency and fuzziness in complete decision tables

In this section, we discuss how to measure the consistency of a target concept and a decision table and establish the relationships between the consistency measure, the fuzziness measure and the inclusion degree in complete decision tables.

Let $S = (U, A)$ be a complete information system, $P, Q \subseteq A$, $U/IND(P) = \{P_1, P_2, \ldots, P_m\}$ and $U/IND(Q) = \{Q_1, Q_2, \ldots, Q_n\}$. We define a partial relation $\preccurlyeq_1$ as follows:

$P \preccurlyeq_1 Q \Leftrightarrow$ for every $P_i \in U/IND(P)$, there exists $Q_j \in U/IND(Q)$ such that $P_i \subseteq Q_j$ [13,12].

If $P \preccurlyeq_1 Q$ and $P \neq Q$, i.e., for some $P_{i_0} \in U/IND(P)$, there exists $Q_{j_0} \in U/IND(Q)$ such that $P_{i_0} \subset Q_{j_0}$, then we denote it as $P \prec_1 Q$. If $P \preccurlyeq_1 Q$, we say that Q is coarser than P (or P is finer than Q). If $P \prec_1 Q$, we say that Q is strictly coarser than P (or P is strictly finer than Q).

Conveniently, by $a(u)$ ($a \in C$) and $d(u)$ ($d \in D$), we denote the values of the object u under condition attribute a and decision attribute d, respectively.

Let $S = (U, C \cup D)$ be a complete decision table, $U/C = \{X_1, X_2, \ldots, X_m\}$ and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$. A condition class $X_i \in U/C$ is said to be consistent if $d(x) = d(y)$, $\forall x, y \in X_i$ and $\forall d \in D$; a decision class $Y_j \in U/D$ is said to be converse consistent if $a(x) = a(y)$, $\forall x, y \in Y_j$ and $\forall a \in C$. It is easy to see that a decision table $S = (U, C \cup D)$ is consistent if every condition class $X_i \in U/C$ is consistent, i.e., $U/C \preccurlyeq_1 U/D$, and S is said to be converse consistent if every decision class $Y_j \in U/D$ is converse consistent, i.e., $U/D \preccurlyeq_1 U/C$. A decision table is called a mixed decision table if it is neither consistent nor converse consistent [25,27].

Definition 1. Let $(X, \leqslant)$ be a poset. A corresponding number $D(y/x)$ ($\forall x, y \in X$) is called the inclusion degree [2]—if the following conditions hold:

(1) $0 \leqslant D(y/x) \leqslant 1$ $(x, y \in X)$;
(2) $x \leqslant y \Rightarrow D(y/x) = 1$ $(x, y \in X)$;
(3) $z \leqslant x \leqslant y \Rightarrow D(z/y) \leqslant D(z/x)$ $(x, y, z \in X)$.

If we modify condition (3) as
(3′) $x \leqslant y \Rightarrow \forall z \in X, D(z/y) \leqslant D(z/x)$ $(x, y \in X)$,
D is called a strong inclusion degree denoted as type S_1. If D is an inclusion degree and further satisfies the condition
(4) $x \leqslant y \Rightarrow \forall z \in X, D(x/z) \leqslant D(y/z)$ $(x, y \in X)$,
then D is a strong inclusion degree denoted as type S_2.

Generally, type S_1 and type S_2 are special cases of inclusion degrees; type S_1 is not all type S_2 and type S_2 is not all type S_1.

Let X, Y be two finite sets. If $X \subseteq Y$, then we say that X is consistent with respect to Y, i.e., X is included in Y. The consistency measure of X with respect to Y can be denoted by $C(X, Y) = \frac{|X \cap Y|}{|X|}$. In fact, $C(X, Y)$ is equivalent to the inclusion degree $D(Y/X) = \frac{|X \cap Y|}{|X|}$ [39].

For arbitrary object $x \in X$, the membership function of x in Y can be denoted by

$$\delta_Y(x) = \begin{cases} \dfrac{|X \cap Y|}{|X|} & \text{if } x \in X \cap Y, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Hence, one can redefine the consistency of X with respect to Y through using the membership function by

$$C(X, Y) = \frac{|X \cap Y|}{|X|} = \frac{1}{|X \cap Y|} \sum_{x \in X} \delta_Y(x), \tag{2}$$

i.e., it can be induced to a fuzzy measure. Obviously, if $X \subseteq Y$, then the consistency of X with respect to Y equals one.

Theorem 1. *Let $(2^U, \subseteq)$ be a poset. Then, $D(Y/X) = \frac{1}{|X \cap Y|} \sum_{x \in X} \delta_Y(x)$ is a type S_2 inclusion degree on the poset $(2^U, \subseteq)$.*

Proof. From $0 \leqslant \delta_Y(x) \leqslant 1$, it follows that $0 \leqslant D(Y/X) \leqslant 1$. When $X \subseteq Y$, one has $X \cap Y = X$ and $\delta_Y(x) = 1$ for arbitrary $x \in X$. Hence, $D(Y/X) = \frac{1}{|X \cap Y|} \sum_{x \in X} \delta_Y(x) = \frac{|X|}{|X|} = 1$.

Let $X \subseteq Y \in 2^U$ and any $Z \in 2^U$. Since

$$D(X/Z) = \frac{1}{|X \cap Z|} \sum_{z \in Z} \delta_X(z),$$

$$D(Y/Z) = \frac{1}{|Y \cap Z|} \sum_{z \in Z} \delta_Y(z),$$

and $X \cap Z \subseteq Y \cap Z$, one has $D(X/Z) = \frac{|X \cap Z|}{|Z|} \leqslant \frac{|Y \cap Z|}{|Z|} = D(Y/Z)$. Therefore, $D(Y/X)$ is a type S_2 inclusion degree on the poset $(2^U, \subseteq)$. $\square$

In [21], the consistency degree of a classification is introduced by Pawlak. Let $F = \{Y_1, Y_2, \ldots, Y_n\}$ be a classification of the universe U and C a condition attribute set. $\underline{C}F = \{\underline{C}Y_1, \underline{C}Y_2, \ldots, \underline{C}Y_n\}$ and $\overline{C}F = \{\overline{C}Y_1, \overline{C}Y_2, \ldots, \overline{C}Y_n\}$ are called C-lower and C-upper approximations of F, where $\underline{C}Y_i = \bigcup\{x \in U | [x]_C \subseteq Y_i \in F\} (1 \leqslant i \leqslant n)$ and $\overline{C}Y_i = \bigcup\{x \in U | [x]_C \cap Y_i \neq \emptyset, Y_i \in F\} (1 \leqslant i \leqslant n)$. The consistency degree of a decision table $S = (U, C \cup D)$, an important measure in rough set theory, is defined as

$$c_C(D) = \frac{\sum_{i=1}^{n} |\underline{C}Y_i|}{|U|}. \tag{3}$$

The consistency degree expresses the percentage of objects which can be correctly classified to decision classes of U/D by the condition attribute set C. In a sense, $c_C(D)$ can be used to measure the consistency of a decision table.

Let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. If we denote by $D(Q/P) = \sum_{Y_i \in U/Q} |\underline{P}Y_i|/|U|$, then $D(Q/P)$ is a type S_1 or type S_2 inclusion degree on the poset $(\mathbf{P}(A), \subseteq)$ [28].

However, the consistency of a decision table cannot be well characterized by this consistency degree when its value achieves zero. As we know, the fact that consistency degree equals zero only implies that the lower approximation of the target decision is an empty set in this decision table. In this situation, the consistency of each of the equivalence classes included in the lower approximation is completely ignored. Nevertheless, these equivalence classes in the lower approximation can be used to extract some uncertain decision rules from this decision table. We cannot omit the consistencies of these equivalence classes in the lower approximation in a decision table with its consistency degree zero. Thus, this measure cannot give elaborate depictions of the consistency of a complete decision table. In the following, we will introduce another measure to assess the consistency of a complete decision table.

At first, we discuss the consistency of an equivalence class X in the condition part U/C in a given complete decision table.

In the rough set literature, rough membership function introduced in [33–35] can be used to measure degrees of inclusion of decision classes into subsets of the universe. Let $S = (U, C \cup D)$ be a complete decision table, $X \in U/C$ an equivalence class and $U/D = \{[u]_D : u \in U\}$. For any object $u \in U$, the membership function of u in X is denoted by

$$\delta_X(u) = \frac{|X \cap [u]_D|}{|X|}, \tag{4}$$

where $\delta_X(u)$ $(0 \leqslant \delta_X(u) \leqslant 1)$ represents a fuzzy concept.

In fact, if $\delta_X(u) = 1$, then X can be said to be consistent with respect to $[u]_D$. In other words, if X is a consistent set with respect to $[u]_D$, then one has $X \subseteq [u]_D$. It can generate a fuzzy set $F_X^D = \{(u, \delta_X(u)) | u \in U\}$ on the universe U. Based on the above membership function [33–35], one can define an inconsistency measure of any subset on the universe related to the decision partition of the decision table as follows.

Definition 2. Let $S = (U, C \cup D)$ be a complete decision table, $X \in U/C$ an equivalence class and $U/D = \{[u]_D : u \in U\}$. An inconsistency measure of X with respect to D is defined as

$$E(F_X^D) = \sum_{i=1}^{|U|} \delta_X(u_i)(1 - \delta_X(u_i)), \tag{5}$$

where $\delta_X(u_i)$ is the membership function of $u_i \in U$ in X.

The class of all fuzzy (crisp, respectively) sets of U is denoted by $F(U)$ ($P(U)$, respectively). For $A \in F(U)$, $u \in U$, $\delta_A(u)$ is the degree of u in A. If $A \in P(U)$, then $A(\cdot)$ expresses the characteristic function of A. Denote by $\underline{\underline{a}}$, $\forall a \in [0, 1]$, the constant fuzzy set with its membership function given by $\underline{\underline{a}}(u) = a$, $\forall u \in U$. In general, the axiomatic definition of a fuzzy entropy on $F(U)$ is given by the following definition.

Definition 3 (*Liang et al. [13], Liang and Li [12], Mi et al. [17]*). A real function $e : F(U) \to [0, 1]$ is referred to as an entropy on $F(U)$ if it satisfies the following conditions:

(1) $e(A) = 0$ iff $A \in P(U)$;
(2) $e(A) = \max_{A \in F(U)} e(A)$ iff $A = \underline{\underline{0.5}}$;
(3) for any $A, B \in F(U)$, if $\delta_B(u) \geqslant \delta_A(u)$ for $\delta_A(u) \geqslant \frac{1}{2}$ or if $\delta_B(u) \leqslant \delta_A(u)$ for $\delta_A(u) \leqslant \frac{1}{2}$, then $e(A) \geqslant e(B)$; and
(4) $e(A) = e(A^c)$, $\forall A \in F(U)$.

Theorem 2. *The inconsistency measure E is an entropy on $F(U)$.*

Proof. By Definition 3, we have that:

(1) If $X \in P(U)$, then, for all $u_i \in U$, either $\delta_X(u_i) = 0$ or $\delta_X(u_i) = 1$. Therefore, $E(X) = 0$. On the other hand, let $E(X) = 0$, then, for all $u_i \in U$, $\delta_X(u_i)(1 - \delta_X(u_i)) = 0$. It follows that either $\delta_X(u_i) = 0$ or $\delta_X(u_i) = 1$, i.e., X is a crisp set.

(2) Since $0 \leqslant \delta_X(u) \leqslant 1$, we have that $\max_{X \in F(U)}(\delta_X(u)(1 - \delta_X(u))) = (\delta_{X_0}(u)(1 - \delta_{X_0}(u))) = \frac{1}{4}$, where $X_0 \in F(U)$, and $\delta_X(u) = \frac{1}{2}$ for any $u \in U$. Hence, $E(\underline{\underline{0.5}}) = \max_{X \in F(U)} E(X)$.

(3) Let $X, Y \in F(U)$. If $\delta_X(u_i) \geqslant \frac{1}{2}$ and $\delta_Y(u_i) \geqslant \delta_X(u_i)$ for all $u_i \in U$, then

$$\begin{aligned} E(X) &= \sum_{i=1}^{|U|} \delta_X(u_i)(1 - \delta_X(u_i)) \\ &= \sum_{i=1}^{|U|} (-(\delta_X(u_i) - 0.5)^2 + 0.25) \\ &= \frac{|U|}{4} - \sum_{i=1}^{|U|} (\delta_X(u_i) - 0.5)^2 \\ &\geqslant \frac{|U|}{4} - \sum_{i=1}^{|U|} (\delta_Y(u_i) - 0.5)^2 \\ &= E(Y). \end{aligned}$$

If $\delta_X(u_i) \leqslant \frac{1}{2}$ and $\delta_Y(u_i) \leqslant \delta_X(u_i)$ for all $u_i \in U$, similar to the above proof, one has $E(X) \geqslant E(Y)$.

(4) $\forall X \in F(U)$, since $\delta_{\sim X}(u_i) = 1 - \delta_X(u_i)$, it follows that for all $u_i \in U$, $\delta_{\sim X}(u_i)(1 - \delta_{\sim X}(u_i)) = (1 - \delta_X(u_i))\delta_X(u_i)$. Therefore, $E(X) = E(\sim X)$.

Summarizing (1)–(4) above, we conclude that the inconsistency measure E is an entropy on $F(U)$. This completes the proof. $\square$

Theorem 3. *The inconsistency measure of a consistent set in a complete decision table is zero.*

Proof. Let $S = (U, C \cup D)$ be a complete decision table, $X \in U/C$ an equivalence class, and $U/D = \{[u]_D : u \in U\}$. If X is a consistent set, then, for any $u \in X$, there exists a decision class $[u]_D$ such that $X \subseteq [u]_D$. So $\delta_X(u) = \frac{|X \cap [u]_D|}{|X|} = \frac{|X|}{|X|} = 1$. For any $u \in U - X$, we have $[u]_D \cap X = \emptyset$, hence $\delta_X(u) = \frac{|X \cap [u]_D|}{|X|} = \frac{|\emptyset|}{|X|} = 0$. Therefore, for $\forall u_i \in U$, one has $\delta_X(u_i)(1 - \delta_X(u_i)) = 0$, i.e., $E(F_X^D) = 0$. Thus, the inconsistency measure of a consistent set in a complete decision table is 0. This completes the proof. $\square$

Definition 4. Let $S = (U, C \cup D)$ be a complete decision table, $X \in U/C$ an equivalence class and $U/D = \{[u]_D : u \in U\}$. A consistency measure of X with respect to D is defined as

$$C(F_X^D) = 1 - \frac{4}{|U|} \sum_{i=1}^{|U|} \delta_X(u_i)(1 - \delta_X(u_i)), \tag{6}$$

where $0 \leqslant C(F_X^D) \leqslant 1$, $\delta_X(u_i)$ is the membership function of $u_i \in U$ in X.

Table 2
A complete decision table about car [8]

Car	Price	Mileage	Size	Max-Speed	d
u_1	High	Low	Full	Low	Good
u_2	Low	High	Full	Low	Good
u_3	Low	Low	Compact	Low	Poor
u_4	High	High	Full	High	Good
u_5	High	High	Full	High	Excellent
u_6	Low	High	Full	Low	Good

Theorem 4. *The consistency measure of a consistent set in a complete decision table is one.*

Proof. The proof is similar to that of Theorem 3. □

In the following, we will investigate the consistency of one partition with respect to another partition in a complete decision table.

Definition 5. Let $S = (U, C \cup D)$ be a complete decision table, $U/C = \{X_1, X_2, \ldots, X_m\}$, and $U/D = \{[u]_D: u \in U\}$. A consistency measure of C with respect to D is defined as

$$C(C, D) = \sum_{j=1}^{m} \frac{|X_j|}{|U|} \left(1 - \frac{4}{|U|} \sum_{i=1}^{|U|} \delta_{X_j}(u_i)(1 - \delta_{X_j}(u_i)) \right), \tag{7}$$

where $\delta_X(u_i)$ is the membership function of $u_i \in U$ in X.

The mechanism of consistency measure is illustrated by the following example.

Example 3. Consider a complete decision table in Table 2, where $C = \{\text{Price, Mileage, Size, Max-Speed}\}$ are the condition attributes and $D = \{\text{d}\}$ is the decision attribute.

By computing, one can obtain that

$U/C = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}\}$ and
$U/d = \{\{u_1, u_2, u_4, u_6\}, \{u_3\}, \{u_5\}\}$.
Let $X_1 = \{u_1\}$, $X_2 = \{u_2, u_6\}$, $X_3 = \{u_3\}$ and $X_4 = \{u_4, u_5\}$. From formula (4), one has that
$\delta_{X_1}(u_1) = \delta_{X_1}(u_2) = \delta_{X_1}(u_4) = \delta_{X_1}(u_6) = 1, \delta_{X_1}(u_3) = \delta_{X_1}(u_5) = 0$;
$\delta_{X_2}(u_1) = \delta_{X_2}(u_2) = \delta_{X_2}(u_4) = \delta_{X_2}(u_6) = 1, \delta_{X_2}(u_3) = \delta_{X_2}(u_5) = 0$;
$\delta_{X_3}(u_3) = 1, \delta_{X_3}(u_1) = \delta_{X_3}(u_2) = \delta_{X_3}(u_4) = \delta_{X_3}(u_5) = \delta_{X_3}(u_6) = 0$ and
$\delta_{X_4}(u_1) = \delta_{X_4}(u_2) = \delta_{X_4}(u_4) = \delta_{X_4}(u_5) = \delta_{X_4}(u_6) = \frac{1}{2}, \delta_{X_3}(u_3) = 0$.

Therefore,

$$\begin{aligned} C(C, D) &= \sum_{j=1}^{4} \frac{|X_j|}{6} \left(1 - \frac{4}{6} \sum_{i=1}^{6} \delta_{X_j}(u_i)(1 - \delta_{X_j}(u_i)) \right) \\ &= \frac{1}{6}(1-0) + \frac{2}{6}(1-0) + \frac{1}{6}(1-0) + \frac{2}{6}\left(1 - \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \times 5\right) \\ &= \frac{13}{18}. \end{aligned}$$

Hence, the consistency measure of C with respect to D in Table 2 is $\frac{13}{18}$.

Theorem 5. *The consistency measure of a consistent complete decision table is one.*

Proof. Let $S=(U, C\cup D)$ be a complete decision table, $U/C=\{X_1, X_2, \ldots, X_m\}$ and $U/D=\{[u]_D : u\in U\}$. If S is consistent, then, for any $X_j\in U/C$, there exists a decision class $[u]_D$ such that $X_j\subseteq [u]_D$. So $\delta_{X_j}(u)=\frac{|X_j\cap[u]_D|}{|X_j|}=\frac{|X_j|}{|X_j|}=1$. For any $u\in U-X_j$, one has $[u]_D\cap X_j=\emptyset$. Hence, $\delta_{X_j}(u)=\frac{|X_j\cap[u]_D|}{|X_j|}=\frac{|\emptyset|}{|X_j|}=0$. Therefore, for $\forall u_i\in U$, one can obtain that $\delta_{X_j}(u_i)(1-\delta_{X_j}(u_i))=0$. Hence,

$$\begin{aligned}C(C,D)&=\sum_{j=1}^{m}\frac{|X_j|}{|U|}\left(1-\frac{4}{|U|}\sum_{i=1}^{|U|}\delta_{X_j}(u_i)(1-\delta_{X_j}(u_i))\right)\\&=\sum_{j=1}^{m}\frac{|X_j|}{|U|}\left(1-\frac{4}{|U|}\times 0\right)\\&=\sum_{j=1}^{m}\frac{|X_j|}{|U|}\\&=1.\end{aligned}$$

Therefore, the consistency measure of a consistent complete decision table is 1. This completes the proof. □

Corollary 1. *If $C(D,C)=1$, then the complete decision table S is converse inconsistent.*

Proof. It can be easily proved from the definition of converse consistency and Definition 5. □

Hence, the consistency of a complete decision table can be measured by using some fuzzy concepts and can be induced to a fuzzy measure.

Theorem 6. *Let $S=(U,A)$ be a complete information system. Then,*

$$D(Q/P)=\sum_{X\in U/P}\frac{|X|}{|U|}\left(1-\frac{4}{|U|}\sum_{i=1}^{|U|}\delta_X(u_i)(1-\delta_X(u_i))\right), \tag{8}$$

where $\delta_X(u_i)=\frac{|X\cap[u_i]_Q|}{|X|}$ with $\delta_X(u_i)\geqslant\frac{1}{2}$, is an inclusion degree on the poset $(\mathbf{P}(A),\preccurlyeq_1)$.

Proof. From the definition of inclusion degree, we have that:

(1) Let $P,Q\in\mathbf{P}(A)$. Then,

$$\begin{aligned}D(Q/P)&=\sum_{X\in U/P}\frac{|X|}{|U|}\left(1-\frac{4}{|U|}\sum_{i=1}^{|U|}\delta_X(u_i)(1-\delta_X(u_i))\right)\\&=\sum_{X\in U/P}\frac{|X|}{|U|}\left(1-\frac{4}{|U|}\sum_{i=1}^{|U|}(-(\delta_X(u_i)-0.5)^2+0.25)\right)\\&=\sum_{X\in U/P}\frac{|X|}{|U|}\cdot\frac{4}{|U|}\sum_{i=1}^{|U|}(\delta_X(u_i)-0.5)^2.\end{aligned}$$

Since $0\leqslant\delta_X(u_i)\leqslant 1$, so $0\leqslant(\delta_X(u_i)-0.5)^2\leqslant\frac{1}{4}$, i.e., $0\leqslant\frac{4}{|U|}\sum_{i=1}^{|U|}(\delta_X(u_i)-0.5)^2\leqslant 1$. Thus, U/P constitutes a partition on U and $0\leqslant D(Q/P)\leqslant 1$.

(2) When $P\preccurlyeq_1 Q$, for $\forall X\in U/P$, there exist some $Y\in U/Q$ such that $X\subseteq Y$. Hence, for $\forall u\in U$, if $u\in Y$, then $\delta_X(u)=\frac{|X\cap[u]_Q|}{|X|}=\frac{|X\cap Y|}{|X|}=\frac{|X|}{|X|}=1$; if $u\notin Y$, then $\delta_X(u)=\frac{|X\cap[u]_Q|}{|X|}=\frac{|\emptyset|}{|X|}=\frac{0}{|X|}=0$. Therefore, $D(Q/P)=\sum_{X\in U/P}\frac{|X|}{|U|}=1$.

(3) Let $P, Q, R \in \mathbf{P}(A)$ with $P \preccurlyeq_1 Q \preccurlyeq_1 R$. Hence, for $\forall [u]_P \in U/P$, there exist some $[u]_Q \in U/Q$ and $[u]_R \in U/R$ such that $[u]_P \subseteq [u]_Q \subseteq [u]_R$. When $\delta_X(u_i) \geqslant 0.5$, one has that

$$\begin{aligned} D(R/P) &= \sum_{X \in U/R} \frac{|X|}{|U|} \cdot \frac{4}{|U|} \sum_{i=1}^{|U|} (\delta_X(u_i) - 0.5)^2 \\ &= \sum_{X \in U/R} \frac{|X|}{|U|} \cdot \frac{4}{|U|} \sum_{i=1}^{|U|} \left(\frac{|X \cap [u]_P|}{|X|} - 0.5 \right)^2 \\ &= \sum_{X \in U/R} \frac{|X|}{|U|} \cdot \frac{4}{|U|} \sum_{i=1}^{|U|} \left(\frac{|[u]_P|}{|X|} - 0.5 \right)^2 \\ &\leqslant \sum_{X \in U/R} \frac{|X|}{|U|} \cdot \frac{4}{|U|} \sum_{i=1}^{|U|} \left(\frac{|[u]_Q|}{|X|} - 0.5 \right)^2 \\ &= D(R/Q). \end{aligned}$$

Therefore, if $\delta_X(u) \geqslant \frac{1}{2}$, $D(Q/P)$ is an inclusion degree on the poset $(\mathbf{P}(A), \preccurlyeq_1)$. This completes the proof. □

As follows, we will research the fuzziness measures of a rough set and a rough decision in a complete decision table.

Given an equivalence relation R on the universe U and a subset $X \subseteq U$, one can define a lower approximation of X and a upper approximation of X by $\underline{R}X = \bigcup\{u \in U | [u]_R \subseteq X\}$ and $\overline{R}X = \bigcup\{u \in U | [u]_R \cap X \neq \emptyset\}$, respectively [21]. The order pair $(\underline{R}X, \overline{R}X)$ is called a rough set of X.

Let $S = (U, A)$ be a complete information system and $X \subseteq U$. For any object $u \in U$, the membership function of u in X is defined as

$$\mu_X(u) = \frac{|X \cap [u]_A|}{|[u]_A|}, \tag{9}$$

where $\mu_X(u) (0 \leqslant \mu_X(u) \leqslant 1)$ represents a fuzzy concept. It can construct a fuzzy set $F_X^A = \{(u, \mu_X(u)) | u \in U\}$ on the universe U.

Definition 6 (*Liang et al. [12]*). Let $S = (U, A)$ be a complete information system and $X \subseteq U$. A fuzziness measure of the rough set X is defined as

$$E(F_X^A) = \sum_{i=1}^{|U|} \mu_X(u_i)(1 - \mu_X(u_i)). \tag{10}$$

Theorem 7 (*Liang et al. [12]*). *In a complete information system $S = (U, A)$, the fuzziness measure of a crisp set equals zero.*

Theorem 8 (*Liang et al. [12]*). *In a complete information system $S = (U, A)$, the fuzziness measure of a rough set is the same as that of its complement set.*

The rough membership function introduced in [33–35] can be used to measure degrees of inclusion of indiscerniblility classes into concepts being approximated. Let $S = (U, A)$ be a complete information system and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$ a target decision. For any $u \in U$, the rough membership function of u in D is defined as [33–35]

$$\mu_D(u) = \frac{|Y_j \cap [u]_A|}{|[u]_A|} \quad (u \in Y_j), \tag{11}$$

where $\mu_D(u) (0 \leqslant \mu_D(u) \leqslant 1)$ denotes a fuzzy concept. It generates a fuzzy set $F_D^A = \{(u, \mu_D(u)) | u \in U\}$ on the universe U. Based on the rough membership function, we will construct a fuzziness measure of a rough decision in the following.

Definition 7 (*Liang et al. [12]*). Let $S = (U, A)$ be a complete information system and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$ a target decision. A fuzziness measure of a rough decision is defined as

$$E(F_D^A) = \sum_{i=1}^{|U|} \mu_D(u_i)(1 - \mu_D(u_i)). \tag{12}$$

In the following example, we show how to calculate the fuzziness measure of a rough decision in a complete information system.

Example 4. (*continued from Example 3*). Let $Y_1 = \{u_1, u_2, u_4, u_6\}$, $Y_2 = \{u_3\}$ and $Y_3 = \{u_5\}$. By computing, one can obtain that

$$\mu_D(u_1) = \mu_D(u_2) = \mu_D(u_3) = \mu_D(u_6) = 1 \quad \text{and} \quad \mu_D(u_4) = \mu_D(u_5) = \tfrac{1}{2}.$$

Therefore,

$$\begin{aligned} E(F_D^A) &= \sum_{i=1}^{6} \mu_D(u_i)(1 - \mu_D(u_i)) \\ &= 1 \times (1 - 1) \times 4 + \frac{1}{2} \times \frac{1}{2} \times 2 \\ &= \frac{1}{2}. \end{aligned}$$

Hence, the fuzziness measure of the rough decision induced by C in Table 2 is $\frac{1}{2}$.

Theorem 9 (*Liang et al. [12]*). *In a complete information system $S = (U, A)$, the fuzziness measure of a crisp decision equals zero.*

Theorem 10. *Let $S = (U, A)$ be a complete information system. Then,*

$$D(Q/P) = 1 - \frac{4}{|U|} \sum_{i=1}^{|U|} \mu_Q(u_i)(1 - \mu_Q(u_i)), \tag{13}$$

where $\mu_Q(u_i) = \frac{|Y_j \cap [u]_P|}{|[u]_P|}$ $(u \in Y_j, Y_j \in U/Q)$ with $\mu_Q(u_i) \geqslant 0.5$, is a type S_2 inclusion degree on the poset $(\mathbf{P}(A), \preccurlyeq_1)$.

Proof. From the definition of inclusion degree, we have that:

(1) Let $P, Q \in \mathbf{P}(A)$. Since $0 \leqslant \mu_Q(u_i) \leqslant 1$, similar to (1) in Theorem 6, one has $0 \leqslant D(Q/P) \leqslant 1$.

(2) When $P \preccurlyeq_1 Q$, for any $X \in U/P$, there exist some $Y \in U/Q$ such that $X \subseteq Y$. Hence, for $\forall u_i \in Y_j$ and $Y_j \in U/Q$, one has that $\mu_Q(u_i) = \frac{|Y_j \cap [u_i]_P|}{|[u_i]_P|} = \frac{|[u_i]_P|}{|[u_i]_P|} = 1$. Therefore, $D(Q/P) = 1 - \frac{4}{|U|} \cdot 0 = 1$.

(3) Let $P, Q, R \in \mathbf{P}(A)$ with $P \preccurlyeq_1 Q$. Hence, for $\forall X \in U/P$, there exist some $Y \in U/Q$ such that $X \subseteq Y$. When $\mu_X(u_i) \geqslant 0.5$ and $\mu_Y(u_i) \geqslant 0.5$, one has that

$$\begin{aligned} D(P/R) &= \frac{4}{|U|} \sum_{i=1, X \in U/P}^{|U|} (\mu_X(u_i) - 0.5)^2 \\ &= \frac{4}{|U|} \sum_{i=1, X \in U/P}^{|U|} \left(\frac{|X \cap [u_i]_R|}{|[u_i]_R|} - 0.5 \right)^2 \\ &\leqslant \frac{4}{|U|} \sum_{i=1, Y \in U/Q}^{|U|} \left(\frac{|Y \cap [u_i]_R|}{|[u_i]_R|} - 0.5 \right)^2 \\ &= \frac{4}{|U|} \sum_{i=1, Y \in U/Q}^{|U|} (\mu_Y(u_i) - 0.5)^2 \\ &= D(Q/R). \end{aligned}$$

Table 3
Values of consistency and fuzziness induced by each condition attribute in Table 2

	Price	Mileage	Size	Max-Speed
Consistency	0.2593	0.3056	0.5556	0.3056
Fuzziness	1.3333	1.2500	0.8000	1.2500

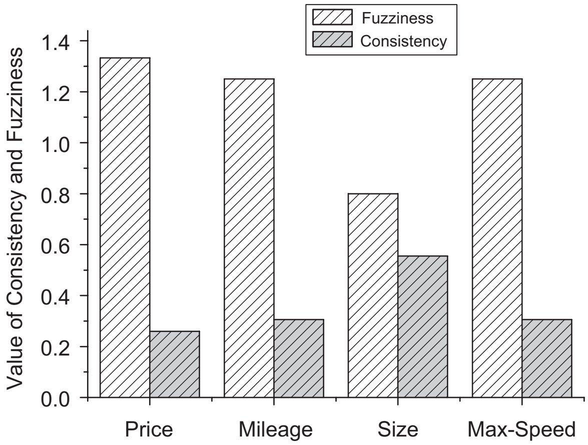

Fig. 1. Fuzziness and consistency induced by each condition attribute in Table 2.

Therefore, if $\mu_X(u) \geqslant \frac{1}{2}$ and $\mu_Y(u) \geqslant \frac{1}{2}$, then $D(Q/P)$ is a type S_2 inclusion degree on the poset $(\mathbf{P}(A), \preccurlyeq_1)$. This completes the proof. $\square$

From Definitions 5 and 7, one can know that the consistency measure denotes the degree of consistency of the condition part with respect to the decision part and the fuzziness measure is the degree of fuzziness of the rough decision approximated by the condition attributes. If we only consider one condition attribute, then the consistency measure and fuzziness measure induced by this attribute can be calculated. In practical decision issues, a decision maker always need to acquire decision rules with much higher consistency and a rough decision with much smaller fuzziness. Hence, the consistency measure and the fuzziness measure can be used to construct heuristic functions for rule extraction and rough decision, respectively.

In the following, through experimental analyses, we illustrate the validity of these two measures for constructing a heuristic function in the complete decision table of Table 2. The values of the consistency and fuzziness induced by each condition attribute of Table 2 are shown in Table 3 and Fig. 1.

It can be seen from Fig. 1 that the consistency measure of the condition attribute Size is the biggest and the consistency measure of the condition attribute Price is the smallest, and the fuzziness measure of the condition attribute Size is the smallest and the consistency measure of the condition attribute Price is the biggest. From Table 3 and Fig. 1, one can get two arrays of these four attributes as follows.

(1) Consistency: Size→Mileage, Max-Speed→Price.
(2) Fuzziness: Size→Mileage, Max-Speed→Price.

The first array can be used to heuristically extract decision rules from a complete decision table, the second array can be used to heuristically obtain the rough decision of a target decision in a complete decision table. Note that these two measures can also be used to evaluate the decision performance of a complete decision table.

4. The consistency and fuzziness in incomplete decision tables

In this section, we introduce the concept of a consistency measure to calculate the degree of the condition part included in the decision part and give a fuzziness measure to compute the fuzziness of a rough set and a rough decision in incomplete decision tables.

Let $S=(U,A)$ be an incomplete information system, $P,Q\subseteq A$, $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$. We define a partial relation $\preccurlyeq_2$ as

$$P\preccurlyeq_2 Q \quad\Leftrightarrow\quad S_P(u_i)\subseteq S_Q(u_i),\qquad \forall i\in\{1,2,\ldots,|U|\}.$$

If $P\preccurlyeq_2 Q$, we say that Q is coarser than P (or P is finer than Q). If $P\preccurlyeq_2 Q$ and $P\neq Q$, we say that Q is strictly coarser than P (or P is strictly finer than Q), denoted by $P\prec_2 Q$. In fact, $P\prec_2 Q\Leftrightarrow \forall i\in\{1,2,\ldots,|U|\}$, one has that $S_P(u_i)\subseteq S_Q(u_i)$, and there exists $j\in\{1,2,\ldots,|U|\}$ such that $S_P(u_j)\subset S_Q(u_j)$.

In [13], we have proved that $(\mathbf{P}(A),\preccurlyeq_2)$ is a poset.

Let $S=(U,C\cup D)$ be an incomplete decision table, $U/SIM(C)=\{S_C(u_1),S_C(u_2),\ldots,S_C(u_{|U|})\}$, $U/SIM(D)=\{S_D(u_1),S_D(u_2),\ldots,S_D(u_{|U|})\}$ and $U/D=\{Y_1,Y_2,\ldots,Y_n\}$. The target decision constitutes a partition on the universe in general. In other words, U/D is equivalent to $U/SIM(D)$ in the essence. Let $Y_j=\{u_{j1},u_{j2},\ldots,u_{js_j}\}$, where $|Y_j|=s_j$ and $\sum_{j=1}^n s_j=|U|$. Then, the relationship between U/D and $U/SIM(D)$ is as follows:

$$Y_j=S_D(u_{j1})=S_D(u_{j2})=\cdots=S_D(u_{js_j}),$$

$$|Y_j|=|S_D(u_{j1})|=|S_D(u_{j2})|=\cdots=|S_D(u_{js_j})|.$$

A condition class $S_C(u)\in U/SIM(C)$ is said to be consistent if $S_C(u)\subseteq S_D(u)$, where $S_D(u)\in U/SIM(D)$; a decision class $S_D(u)\in U/SIM(D)$ is said to be converse consistent if $S_D(u)\subseteq S_C(u)$. For an incomplete decision table, it is easy to see that a decision table $S=(U,C\cup D)$ is consistent if every condition class $S_C(u_i)\in U/SIM(C)$ is consistent, i.e., $C\preccurlyeq_2 D$ [40]; S is said to be converse consistent if every decision class $S_D(u_i)\in U/SIM(D)$ is converse consistent, i.e., $D\preccurlyeq_2 C$. An incomplete decision table is called an incomplete mixed decision table if it is neither consistent nor converse consistent.

Let $F=U/D=\{Y_1,Y_2,\ldots,Y_n\}$ be a classification of the universe U and C a condition attribute set. $\underline{SIM(C)}(F)=\{\underline{SIM(C)}(Y_1),\underline{SIM(C)}(Y_2),\ldots,\underline{SIM(C)}(Y_n)\}$ and $\overline{C}F=\{\overline{SIM(C)}(Y_1),\overline{SIM(C)}(Y_2),\ldots,\overline{SIM(C)}(Y_n)\}$ are called C-lower and C-upper approximations of F, where $\underline{SIM(C)}(Y_i)=\bigcup\{u\in U|S_C(u)\subseteq Y_i\in F\}$ $(1\leqslant i\leqslant n)$ and $\overline{SIM(C)}(Y_i)=\bigcup\{u\in U|S_C(u)\cap Y_i\neq\emptyset, Y_i\in F\}$ $(1\leqslant i\leqslant n)$. Naturally, the consistency degree of an incomplete decision table $S=(U,C\cup D)$ can also be defined as

$$c_C(D)=\frac{\sum_{i=1}^n|\underline{SIM(C)}(Y_i)|}{|U|}. \tag{14}$$

The consistency degree expresses the percentage of objects which can be correctly classified to the decision classes of U/D by the condition attribute set C. In a sense, $c_C(D)$ can be used to measure the consistency of a decision table. It should be pointed out that this measure is not a fuzzy measure.

However, the consistency of an incomplete decision table cannot be well characterized by this consistency degree because it only takes into account that the lower approximation of a target decision. Therefore, we will introduce another measure to assess the consistency of an incomplete decision table.

Firstly, we investigate the consistency of the tolerance class $S_C(u_i)$ $(i\in\{1,2,\ldots,|U|\})$ included the condition part $U/SIM(C)$ in an incomplete decision table.

Let $S=(U,C\cup D)$ be an incomplete decision table, $S_C(u_i)\in U/SIM(C)$ a tolerance class, $U/D=\{[u]_D:u\in U\}$ and $U/SIM(D)=\{S_D(u):u\in U\}$. For any object $u\in U$, the membership function of u in the tolerance class $S_C(u_i)$ is defined as

$$\delta_{S_C(u_i)}(u)=\begin{cases}\dfrac{|S_C(u_i)\cap S_D(u)|}{|S_C(u_i)|} & \text{if } u=u_i,\\ 0 & \text{if } u\neq u_i,\end{cases} \tag{15}$$

where $\delta_{S_C(u_i)}(u)$ $(0\leqslant\delta_{S_C(u_i)}(u)\leqslant 1)$ denotes a fuzzy concept.

If $\delta_{S_C(u_i)}(u)=1$, then the tolerance class $S_C(u_i)$ can be said to be consistent with respect to D. In other words, if $S_C(u_i)$ is a consistent set with respect to D, then $S_C(u_i)\subseteq S_D(u_i)$. It can generate a fuzzy set $F^D_{S_C(u_i)}=\{(u,\delta_{S_C(u_i)}(u))|u\in U\}$ on the universe U.

Definition 8. Let $S = (U, C \cup D)$ be an incomplete decision table, $S_C(u_i) \in U/SIM(C)$ a tolerance class and $U/SIM(D) = \{S_D(u) : u \in U\}$. A consistency measure of $S_C(u_i)$ with respect to D is defined as

$$C(F^D_{S_C(u_i)}) = \sum_{u \in U} \delta_{S_C(u_i)}(u), \tag{16}$$

where $0 \leqslant C(F^D_{S_C(u_i)}) \leqslant 1$.

Theorem 11. *The consistency measure of a consistent tolerance class in an incomplete decision table is one.*

Proof. It is straightforward. □

In the following, based on the above discussion, we research the consistency between two attribute subsets in an incomplete decision table.

Definition 9. Let $S = (U, C \cup D)$ be an incomplete decision table, $U/SIM(C) = \{S_C(u_1), S_C(u_2), \ldots, S_C(u_{|U|})\}$ and $U/SIM(D) = \{S_D(u) : u \in U\}$. A consistency measure of C with respect to D is defined as

$$C(C, D) = \frac{1}{|U|} \sum_{i=1}^{|U|} \sum_{u \in U} \delta_{S_C(u_i)}(u), \tag{17}$$

where $0 \leqslant C(C, D) \leqslant 1$ and $\delta_{S_C(u_i)}(u)$ is the membership function of $u \in U$ in the tolerance class $S_C(u_i)$.

This definition is illustrated by the following example.

Example 5. (*continued from Example 1*). From Table 1, we know that

$$U/SIM(C) = \{S_C(u_1), S_C(u_2), S_C(u_3), S_C(u_4), S_C(u_5), S_C(u_6)\},$$

where $S_C(u_1) = \{u_1\}$, $S_C(u_2) = \{u_2, u_6\}$, $S_C(u_3) = \{u_3\}$, $S_C(u_4) = \{u_4, u_5\}$, $S_C(u_5) = \{u_4, u_5, u_6\}$ and $S_C(u_6) = \{u_2, u_5, u_6\}$.

From formula (15), one has that
$\delta_{S_C(u_1)}(u_1) = 1$, $\delta_{S_C(u_i)}(u_1) = 0$ $(i \neq 1)$; $\delta_{S_C(u_2)}(u_2) = 1$, $\delta_{S_C(u_i)}(u_2) = 0$ $(i \neq 2)$;
$\delta_{S_C(u_3)}(u_3) = 1$, $\delta_{S_C(u_i)}(u_3) = 0$ $(i \neq 3)$; $\delta_{S_C(u_4)}(u_4) = \frac{1}{2}$, $\delta_{S_C(u_i)}(u_4) = 0$ $(i \neq 4)$;
$\delta_{S_C(u_5)}(u_5) = \frac{1}{3}$, $\delta_{S_C(u_i)}(u_5) = 0$ $(i \neq 5)$; and $\delta_{S_C(u_6)}(u_6) = \frac{2}{3}$, $\delta_{S_C(u_i)}(u_6) = 0$ $(i \neq 6)$.

Therefore,

$$\begin{aligned} C(C, D) &= \frac{1}{6} \sum_{i=1}^{6} \sum_{u \in U} \delta_{S_C(u_i)}(u) \\ &= \frac{1}{6}\left(1 + 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{2}{3}\right) \\ &= \frac{3}{4}. \end{aligned}$$

Hence, the consistency measure of C with respect to D in Table 1 is $\frac{3}{4}$.

Theorem 12. *The consistency measure of a consistent incomplete decision table is one.*

Proof. Let $S = (U, C \cup D)$ be an incomplete decision table, $U/SIM(C) = \{S_C(u_1), S_C(u_2), \ldots, S_C(u_{|U|})\}$ and $U/SIM(D) = \{S_D(u) : u \in U\}$. If S is consistent, then, for any $u_i \in U$, one has $S_C(u_i) \subseteq S_D(u_i)$. Hence, when $u = u_i$, we have $\delta_{S_C(u_i)}(u) = \frac{|S_C(u_i) \cap S_D(u_i)|}{|S_C(u_i)|} = \frac{|S_C(u_i)|}{|S_C(u_i)|} = 1$; otherwise, $\delta_{S_C(u_i)}(u) = 0$. Therefore,

$$\begin{aligned} C(C, D) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \sum_{u \in U} \delta_{S_C(u_i)}(u) \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} (1 \cdot 1 + (|U| - 1) \cdot 0) \\ &= 1. \end{aligned}$$

Thus the consistency measure of a consistent incomplete decision table is 1. This completes the proof. □

Corollary 2. *If $C(D, C) = 1$, then the incomplete decision table S is converse consistent.*

Proof. It can be proved from the definition of converse consistency in incomplete decision tables and Definition 9. □

Consequently, the consistency of an incomplete decision table can be measured through using some fuzzy concepts and it can also be induced to a fuzzy measure.

Theorem 13. *Let $S = (U, A)$ be an incomplete information system. Then,*

$$D(Q/P) = \frac{1}{|U|} \sum_{i=1}^{|U|} \sum_{u \in U} \delta_{S_P(u_i)}(u) \tag{18}$$

is a type S_2 inclusion degree on the poset $(\mathbf{P}(A), \preccurlyeq_2)$.

Proof. From the definition of inclusion degree, we have that:

(1) Let $P, Q \in \mathbf{P}(A)$. From $0 \leqslant \frac{|S_P(u_i) \cap S_Q(u)|}{|S_P(u_i)|} \leqslant 1$, it follows that $\delta_{S_P(u_i)}(u) = 0$ $(u \neq u_i)$ or $\delta_{S_P(u_i)}(u) = \frac{|S_P(u_i) \cap S_Q(u)|}{|S_P(u_i)|}$ $(u = u_i)$. Hence, $0 \leqslant \sum_{u \in U} \delta_{S_P(u_i)}(u) \leqslant 1$. Thus, $0 \leqslant D(Q/P) \leqslant 1$.

(2) When $P \preccurlyeq_2 Q$, one can obtain that $S_P(u_i) \subseteq S_Q(u_i)$, $\forall i \in \{1, 2, \ldots, |U|\}$. So, for any $i \leqslant |U|$, if $u = u_i$, then $\delta_{S_P(u_i)}(u) = 1$; otherwise $\delta_{S_P(u_i)}(u) = 0$. Hence, $\sum_{u \in U} \delta_{S_P(u_i)}(u) = 1 + (|U| - 1) \cdot 0 = 1$. Therefore, one has that $D(Q/P) = \frac{1}{|U|} \sum_{i=1}^{|U|} 1 = 1$.

(3) Let $P, Q, R \in \mathbf{P}(A)$ with $P \preccurlyeq_2 Q$. Hence, it follows that $S_P(u_i) \subseteq S_Q(u_i)$, $\forall i \in \{1, 2, \ldots, |U|\}$. Thus,

$$\begin{aligned} D(P/R) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \sum_{u \in U} \delta_{S_R(u_i)}(u) \\ &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_R(u_i) \cap S_P(u_i)|}{|S_R(u_i)|} \\ &\leqslant \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|S_R(u_i) \cap S_Q(u_i)|}{|S_R(u_i)|} \\ &= D(Q/R). \end{aligned}$$

Therefore, $D(Q/P)$ is a type S_2 inclusion degree on the poset $(\mathbf{P}(A), \preccurlyeq_2)$. This completes the proof. □

In succession, we discuss the fuzziness measure of a rough set and a rough decision in incomplete information systems.

Let $S = (U, A)$ be an incomplete information system and $X \subseteq U$. One can define a lower approximation of X and an upper approximation of X by $\underline{SIM(A)}(X) = \bigcup\{u \in U | S_A(u) \subseteq X\}$ and $\overline{SIM(A)}(X) = \bigcup\{u \in U | S_A(u) \cap X \neq \emptyset\}$, respectively [8]. The order pair $(\underline{SIM(A)}(X), \overline{SIM(A)}(X))$ is called a rough set of X with respect to A in the context of incomplete information systems.

Let $S = (U, A)$ be an incomplete information system and $X \subseteq U$. For any object $u \in U$, the membership function of u in X is defined as

$$\mu_X(u) = \frac{|X \cap S_A(u)|}{|S_A(u)|}, \tag{19}$$

where $\mu_X(u)$ $(0 \leqslant \mu_X(u) \leqslant 1)$ represents a fuzzy concept. It can construct a fuzzy set $F_X^A = \{(u, \mu_X(u)|u \in U\}$ on the universe U.

Definition 10. Let $S = (U, A)$ be an incomplete information system and $X \subseteq U$, a fuzziness measure of the rough set X is defined as

$$E(F_X^A) = \sum_{i=1}^{|U|} \mu_X(u)(1 - \mu_X(u)), \tag{20}$$

where $\mu_X(u) = \frac{|X \cap S_A(u)|}{|S_A(u)|}$, $S_A(u) \in U/SIM(A)$.

Theorem 14. *In an incomplete information system $S = (U, A)$, the fuzziness measure of a crisp set equals zero.*

Proof. Let X be a crisp set in the incomplete information system $S = (U, A)$, then $\underline{SIM(A)}(X) = X = \overline{SIM(A)}(X)$. Hence, for any $u \in U$, if $S_A(u) \subseteq X$, then $\mu_X(u) = 1$, and if $S_A(u) \nsubseteq X$, then $\mu_X(u) = 0$. Thus, for $\forall u \in U$, one has that $\mu_X(u)(1 - \mu_X(u)) = 0$, i.e., $E(F_X^A) = 0$. This completes the proof. □

Theorem 15. *In an incomplete information system $S = (U, A)$, the fuzziness measure of a rough set is the same as that of its complement set.*

Proof. Let X be a rough set in the incomplete information system $S = (U, A)$ and X^c is its complement set on the universe U, i.e., $X^c = U - X$. For any $u \in U$, one has that

$$\mu_X(u) + \mu_{X^c}(u) = \frac{|X \cap S_A(u)|}{|S_A(u)|} + \frac{|X^c \cap S_A(u)|}{|S_A(u)|} = \frac{|S_A(u)|}{|S_A(u)|} = 1,$$

i.e., $\mu_{X^c}(u) = 1 - \mu_X(u)$. Thus, for any $u \in U$, one can obtain that $\mu_X(u)(1 - \mu_X(u)) = \mu_{X^c}(u)(1 - \mu_{X^c}(u))$, i.e., $E(F_X^A) = E(F_{X^c}^A)$. This completes the proof. □

Let $S = (U, C \cup D)$ an incomplete decision table and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$ a target decision. For any $u \in U$, the membership function of u in D is defined as

$$\mu_D(u) = \frac{|Y_j \cap S_C(u)|}{|S_C(u)|} \quad (u \in Y_j), \tag{21}$$

where $\mu_D(u)$ $(0 \leqslant \mu_D(u) \leqslant 1)$ represents a fuzzy concept. It generates a fuzzy set $F_D^C = \{(u, \mu_D(u))|u \in U\}$ on the universe U.

Definition 11. Let $S = (U, C \cup D)$ an incomplete decision table, $U/D = \{Y_1, Y_2, \ldots, Y_n\}$ a target decision, a fuzziness measure of a rough decision is defined as

$$E(F_D^C) = \sum_{i=1}^{|U|} \mu_D(u_i)(1 - \mu_D(u_i)), \tag{22}$$

where $\mu_D(u) = \frac{|Y_j \cap S_C(u)|}{|S_C(u)|}$ $(u \in Y_j)$.

Table 4
Values of consistency and fuzziness induced by each condition attribute in Table 1

	Price	Mileage	Size	Max-Speed
Consistency	0.3889	0.4778	0.7333	0.5694
Fuzziness	1.2778	1.2022	0.8000	1.2292

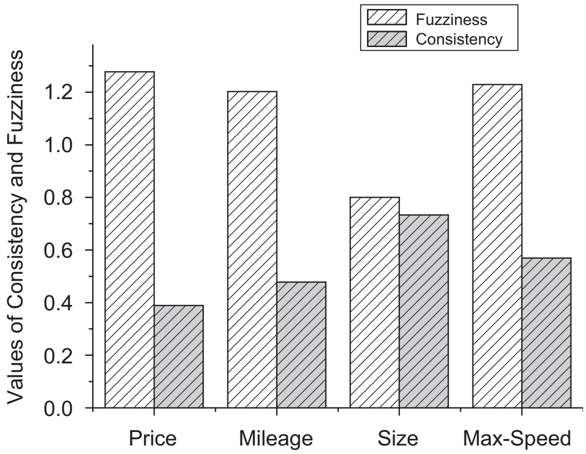

Fig. 2. Fuzziness and consistency induced by each condition attribute in Table 1.

In the following example, we show how to compute the fuzziness of a rough decision in an incomplete information system.

Example 6. (*continued from Example 5*). Suppose that $Y_1 = \{u_1, u_2, u_4, u_6\}$, $Y_2 = \{u_3\}$ and $Y_3 = \{u_5\}$.

From formula (21), one can obtain that $\mu_D(u_1) = \mu_D(u_2) = \mu_D(u_3) = 1$, $\mu_D(u_4) = \frac{1}{2}$, $\mu_D(u_5) = \frac{1}{3}$ and $\mu_D(u_6) = \frac{2}{3}$. Therefore,

$$\begin{aligned} E(F_D^C) &= \sum_{i=1}^{6} \mu_D(u_i)(1 - \mu_D(u_i)) \\ &= 1 \times (1-1) \times 3 + \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} \\ &= \frac{25}{36}. \end{aligned}$$

Hence, the fuzziness of the rough decision induced by C in Table 1 is $\frac{25}{36}$.

Theorem 16. *In an incomplete decision table $S = (U, C \cup D)$, the fuzziness measure of a crisp decision equals zero.*

Proof. Let $U/D = \{Y_1, Y_2, \ldots, Y_n\}$ be a crisp decision, i.e., $\underline{SIM(C)}(Y_i) = \overline{SIM(C)}(Y_i)$, $i = \{1, 2, \ldots, n\}$. For any $u \in U$, there exists some $Y_j \in U/D$ such that $u \in Y_j$ and $S_C(u) \subseteq Y_j$. Hence, one has that

$$\mu_D(u) = \frac{|Y_j \cap S_C(u)|}{|S_C(u)|} = \frac{|S_C(u)|}{|S_C(u)|} = 1, \text{ i.e., } 1 - \mu_D(u) = 0.$$

Therefore, $E(F_D^C) = 0$. This completes the proof. $\square$

As follows, through experimental analyses, we illustrate the validity of these two measures for constructing a heuristic function in the incomplete decision table of Table 1. The values of the consistency and fuzziness induced by each condition attribute of Table 1 are shown in Table 4 and Fig. 2.

From Table 4 and Fig. 2, it is easy to see that the consistency measure of the condition attribute Size is the biggest and the consistency measure of the condition attribute Price is the smallest, and the fuzziness measure of the condition attribute Size is the smallest and the consistency measure of the condition attribute Price is the biggest. According to their values, one can obtain the following two arrays of these four attributes.

(1) Consistency: Size→Max-Speed→ Mileage→Price.
(2) Fuzziness: Size→Mileage→Max-Speed→Price.

The first array can be used to heuristically extract decision rules from an incomplete decision table, the second array can be used to heuristically obtain the rough decision of a target decision in an incomplete decision table. Clearly, these two measures are also suitable for assessing the decision performance of an incomplete decision table.

5. The consistency and fuzziness in the context of maximal consistent blocks

As we know, tolerance classes are not minimal units for describing knowledge or information in incomplete information systems and incomplete decision tables [4,10]. In this section, a consistency measure and a fuzziness measure in the context of maximal consistent blocks are introduced to an incomplete decision table and their several properties are obtained.

At first, we define another partial relation in incomplete information systems. Let $S = (U, A)$ be an incomplete information system, $P, Q \subseteq A$, $MC_P = \{P^1, P^2, \ldots, P^m\}$ and $MC_Q = \{Q^1, Q^2, \ldots, Q^n\}$. We define a partial relation $\preccurlyeq_3$ as follows:

$P \preccurlyeq_3 Q \Leftrightarrow$ for every $P^i \in MC_P$, there exists $Q^j \in MC_Q$ such that $P^i \subseteq Q^j$.

If $P \preccurlyeq_3 Q$ and $P \neq Q$, i.e., for some $P^{i_0} \in MC_P$, there exists $Q^{j_0} \in MC_Q$ such that $P^{i_0} \subset Q^{j_0}$, then we denote it as $P \prec_3 Q$.

Theorem 17. $(\mathbf{P}(A), \preccurlyeq_3)$ *is a poset.*

Proof. Let $S = (U, A)$ be an incomplete information system, $P, Q \subseteq A$, $MC_P = \{P^1, P^2, \ldots, P^m\}$, $MC_Q = \{Q^1, Q^2, \ldots, Q^n\}$ and $MC_R = \{R^1, R^2, \ldots, R^l\}$.

(1) For any $u \in U$, it is obvious that $P^i \subseteq P^i$. Hence, $P \preccurlyeq_3 P$.

(2) Suppose $P \preccurlyeq_3 Q$ and $Q \preccurlyeq_3 P$. When $P \preccurlyeq_3 Q$, it follows from the definition of $\preccurlyeq_3$ that for any $P^i \in MC_P$, there exists $Q^j \in MC_Q$ such that $P^i \subseteq Q^j$.

Next, we prove that $S_P(u) \subseteq S_Q(u)$, $\forall u \in U$. Assume that $MC_P(u) = \{X_1, X_2, \ldots, X_s\}$ and $MC_Q(u) = \{Y_1, Y_2, \ldots, Y_t\}$. We know that $S_P(u) = \bigcup\{X_k \in MC_P | X_k \subseteq S_P(u)\} = \bigcup\{X_k \in MC_P(u)\}$ $(k \leqslant s)$ and $S_Q(u) = \bigcup\{Y_k \in MC_Q | Y_k \subseteq S_Q(u)\} = \bigcup\{Y_k \in MC_Q(u)\}$ $(k \leqslant t)$ from Property 4 in the literature [10]. From the definition of maximal consistent blocks, one has that $u \in MC_P(u)$, $u \in MC_Q(u)$, $u \notin MC_P - MC_P(u)$ and $u \notin MC_Q - MC_Q(u)$. Hence, it follows from $P \preccurlyeq_3 Q$ that for any $X \in MC_P(u)$, there exist $Y \in MC_Q(u)$ such that $X \subseteq Y$. Thus, for any $u \in U$, we have that

$$\begin{aligned} S_P(u) &= \bigcup\{X_k \in MC_P | X_k \subseteq S_P(u)\} = \bigcup_{k=1}^{s} X_k \\ &\subseteq \bigcup_{k=1}^{t} Y_k = \bigcup\{Y_k \in MC_Q | Y_K \subseteq S_Q(u)\} \\ &= S_Q(u), \end{aligned}$$

i.e., $S_P(u) \subseteq S_Q(u)$ holds.

Similarly, if $Q \preccurlyeq_3 P$, then $S_Q(u) \subseteq S_P(u)$, $u \in U$.

Therefore, for any $u \in U$, one has that $S_P(u) \subseteq S_Q(u) \subseteq S_P(u)$. So, $S_P(u) = S_Q(u)$, $\forall u \in U$. Hence, $P = Q$ if $P \preccurlyeq_3 Q$ and $Q \preccurlyeq_3 P$.

(3) Suppose $P \preccurlyeq_3 Q$ and $Q \preccurlyeq_3 R$. When $P \preccurlyeq_3 Q$, it follows from the definition of $\preccurlyeq_3$ that for any $P^i \in MC_P$, there exists $Q^j \in MC_Q$ such that $P^i \subseteq Q^j$. When $Q \preccurlyeq_3 R$, it follows from the definition of $\preccurlyeq_3$ that for any $Q^j \in MC_Q$, there exists $R^k \in MC_R$ such that $Q^j \subseteq R^k$. In other words, for any $P^i \in MC_P$, there exists $R^k \in MC_R$ such that $P^i \subseteq R^k$, i.e., $P \preccurlyeq_3 R$.

Hence, $(\mathbf{P}(A), \preccurlyeq_3)$ forms a poset. This completes the proof. □

As follows, we introduce several new concepts and notations, which will be applied in our further developments.

Definition 12. Let $S = (U, C \cup D)$ be an incomplete decision table, $MC_C = \{X_1, X_2, \ldots, X_m\}$ and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$. In the context of maximal consistent blocks, a maximal consistent block $X_i \in MC_C$ is said to be consistent if $d(u) = d(v), \forall u, v \in X_i, \forall d \in D$; a decision class $Y_j \in U/D$ is said to be converse consistent if there exists a maximal consistent block X_i such that $u, v \in X_i, \forall u, v \in Y_j$.

Definition 13. Let $S = (U, C \cup D)$ be an incomplete decision table, $MC_C = \{X_1, X_2, \ldots, X_m\}$ and $U/D = \{Y_1, Y_2, \ldots, Y_n\}$. In the context of maximal consistent blocks, S is said to be consistent if every maximal consistent block $X_i \in MC_C$ is consistent; S is said to be converse consistent if every decision class $Y_j \in U/D$ is converse consistent. An incomplete decision table is called a mixed decision table if it is neither consistent nor converse consistent.

From the above definitions, in the context of maximal consistent blocks, it follows that:

- an incomplete decision table S is consistent $\Leftrightarrow MC_C \preccurlyeq' MC_D$ $(MC_D = U/D)$,
- an incomplete decision table S is converse consistent $\Leftrightarrow MC_D \preccurlyeq' MC_C$.

In particular, $S = (U, C \cup D)$ is said to be restrict consistent (restrict converse consistent) if $MC_C \prec' MC_D$ $(MC_D \prec' MC_C)$, where $MC_D = U/D$.

Remark. It is deserved to point out that these definitions are natural generalizations of Definitions 2 and 3 in a complete decision table in [25,27]. That is to say, if S is a complete decision table, then the maximal consistent blocks induced by the condition attribute set C will degenerate into the partition induced by C and the partial relation $\preccurlyeq_3$ will degenerate into the partial relation $\preccurlyeq_1$ on all partitions induced by the power set $\mathbf{P}(C)$.

Definition 14 (*Leung and Li [10]*). Let $S = (U, A)$ be an incomplete information system and $P \subseteq A$. The approximation operators $\underline{apr}_P$ and $\overline{apr}_P$ are defined as

$$\underline{apr}_P(X) = \bigcup \{Y \in MC_P | Y \subseteq X\} \quad \text{and}$$

$$\overline{apr}_P(X) = \bigcup \{Y \in MC_P | Y \cap X \neq \emptyset\}.$$

Let $F = U/D = \{Y_1, Y_2, \ldots, Y_n\}$ be a classification of the universe U and C a condition attribute set. In the view of maximal consistent block technique, one can call $\underline{apr}_C F = \{\underline{apr}_C(Y_1), \underline{apr}_C(Y_2), \ldots, \underline{apr}_C(Y_n)\}$ and $\overline{apr}_C F = \{\overline{apr}_C(Y_1), \overline{apr}_C(Y_2), \ldots, \overline{apr}_C(Y_n)\}$ C-lower and C-upper approximations of F, respectively, where $\underline{apr}_C(Y_i) = \bigcup \{u \in U | MC_C(u) \subseteq Y_i, Y_i \in F\}$ $(1 \leqslant i \leqslant n)$ and $\overline{apr}_C Y_i = \bigcup \{u \in U | MC_C(u) \cap Y_i \neq \emptyset, Y_i \in F\}$ $(1 \leqslant i \leqslant n)$.

For an incomplete decision table, one can extend the consistency degree to measure entire consistency of an incomplete decision table. Similar to formula (3), the consistency degree of an incomplete decision table can be defined as

$$c_C(D) = \frac{\sum_{i=1}^{n} |\underline{apr}_C(Y_i)|}{|U|}. \tag{23}$$

Similar to formulae (3) and (14), however, the consistency of an incomplete decision table cannot be well characterized by this consistency degree. It implies that the measure $c_C(D)$ cannot depict the consistency of an incomplete decision table when $c_C(D) = 0$.

Now we will investigate how to measure entire consistency of an incomplete decision table in the context of maximal consistent block technique. Simply, we first discuss the consistency of a maximal consistent block X in the condition part of a given incomplete decision table.

Let $S = (U, C \cup D)$ be an incomplete decision table, $X \in MC_C$ a maximal consistent block and $MC_D = U/D = \{[u]_D : u \in U\}$. For any object $u \in U$, the membership function of u in X is defined as

$$\delta_X(u) = \frac{|X \cap [u]_D|}{|X|},$$

where $\delta_X(u)$ $(0 \leqslant \delta_X(u) \leqslant 1)$ represents a fuzzy concept.

In the view of maximal consistent block technique, if $\delta_X(u) = 1$, then X can be said to be consistent with respect to $[u]_D$. In other words, if X is a consistent set with respect to $[u]_D$, then $X \subseteq [u]_D$. It can generate a fuzzy set $F_X^D = \{(u, \delta_X(u)) | u \in U\}$ on the universe U.

Definition 15. Let $S = (U, C \cup D)$ be an incomplete decision table, $X \in MC_C$ a maximal consistent block and $MC_D = U/D = \{[u]_D : u \in U\}$. A consistency measure of X with respect to D is defined as

$$C(F_X^D) = 1 - \frac{4}{|U|} \sum_{i=1}^{|U|} \delta_X(u_i)(1 - \delta_X(u_i)), \tag{24}$$

where $0 \leqslant C(F_X^D) \leqslant 1$, $\delta_X(u_i)$ is the membership function of $u_i \in U$ in X.

Theorem 18. *The consistency measure of a consistent maximal consistent block is one.*

Proof. Let $S = (U, C \cup D)$ be an incomplete decision table, $X \in MC_C$ a maximal consistent block and $MC_D = U/D = \{[u]_D : u \in U\}$. If X is a consistent set, then, for any $u \in X$, there exists a decision class $[u]_D$ such that $X \subseteq [u]_D$. So, $\delta_X(u) = \frac{|X \cap [u]_D|}{|X|} = \frac{|X|}{|X|} = 1$. And, for any $u \in U - X$, one has $[u]_D \cap X = \emptyset$. Hence, $\delta_X(u) = \frac{|X \cap [u]_D|}{|X|} = \frac{|\emptyset|}{|X|} = 0$. Therefore, for $\forall u_i \in U$, $\delta_X(u_i)(1 - \delta_X(u_i)) = 0$, i.e., $E(F_X^D) = 0$. Thus, the inconsistency measure of a consistent set is 0. This completes the proof. $\square$

In the following, we will research entire consistency of an incomplete decision table in the context of maximal consistent blocks.

Definition 16. Let $S = (U, C \cup D)$ be an incomplete decision table, $MC_C = \{X_1, X_2, \ldots, X_m\}$ and $MC_D = U/D = \{[u]_D : u \in U\}$. A consistency measure of C with respect to D is defined as

$$C(C, D) = \frac{1}{m} \sum_{j=1}^{m} \left(1 - \frac{4}{|U|} \sum_{i=1}^{|U|} \delta_{X_j}(u_i)(1 - \delta_{X_j}(u_i))\right), \tag{25}$$

where $\delta_{X_j}(u_i) = \frac{|X_j \cap [u_i]_D|}{|X_j|}$ is the membership function of $u_i \in U$ in X_j.

Obviously, $0 \leqslant C(C, D) \leqslant 1$. The mechanism of this definition is illustrated by the following example.

Example 7. (*continued from Example 2*). From Table 1, it follows that

$$MC_C = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\}.$$

Let $X_1 = \{u_1\}$, $X_2 = \{u_2, u_6\}$, $X_3 = \{u_3\}$, $X_4 = \{u_4, u_5\}$ and $X_5 = \{u_5, u_6\}$, one can obtain that
$\delta_{X_1}(u_1) = \delta_{X_1}(u_2) = \delta_{X_1}(u_4) = \delta_{X_1}(u_6) = 1$, $\delta_{X_1}(u_3) = \delta_{X_1}(u_5) = 0$;
$\delta_{X_2}(u_1) = \delta_{X_2}(u_2) = \delta_{X_2}(u_4) = \delta_{X_2}(u_6) = 1$, $\delta_{X_2}(u_3) = \delta_{X_2}(u_5) = 0$;
$\delta_{X_3}(u_3) = 1$, $\delta_{X_3}(u_1) = \delta_{X_3}(u_2) = \delta_{X_3}(u_4) = \delta_{X_3}(u_5) = \delta_{X_3}(u_6) = 0$;
$\delta_{X_4}(u_1) = \delta_{X_4}(u_2) = \delta_{X_4}(u_4) = \delta_{X_4}(u_5) = \delta_{X_4}(u_6) = \frac{1}{2}$, $\delta_{X_4}(u_3) = 0$; and
$\delta_{X_5}(u_1) = \delta_{X_5}(u_2) = \delta_{X_5}(u_4) = \delta_{X_5}(u_5) = \delta_{X_5}(u_6) = \frac{1}{2}$, $\delta_{X_5}(u_3) = 0$.
Therefore,

$$\begin{aligned} C(C, D) &= \frac{1}{5} \sum_{j=1}^{5} \left(1 - \frac{4}{|U|} \sum_{i=1}^{6} \delta_{X_j}(u_i)(1 - \delta_{X_j}(u_i))\right) \\ &= \frac{1}{5}\left[(1-0) + (1-0) + (1-0) + \left(1 - \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \times 5\right) + \left(1 - \frac{2}{3} \times \frac{1}{2} \times \frac{1}{2} \times 5\right)\right] \\ &= \frac{2}{3}. \end{aligned}$$

Hence, the consistency measure of C with respect to D in Table 1 in the context of maximal consistent blocks is $\frac{2}{3}$.

Theorem 19. *The consistency measure of a consistent incomplete decision table in the context of maximal consistent blocks is one.*

Proof. The proof is similar to that of Theorem 18. □

In the following, we deal with the fuzziness measure of a rough set and a rough decision in an incomplete decision table in the context of maximal consistent blocks.

Let $S=(U,A)$ be an incomplete information system and $X \subseteq U$. For any object $u \in U$, in the context of maximal consistent blocks, the membership function of u in X is defined as

$$\mu_X(u)=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|X\cap Y|}{|Y|}, \tag{26}$$

where $\mu_X(u)$ $(0\leqslant\mu_X(u)\leqslant 1)$ represents a fuzzy concept. It can generate a fuzzy set $F_X^A=\{(u,\mu_X(u))|u\in U\}$ on the universe U.

Definition 17. Let $S=(U,A)$ be an incomplete information system and $X\subseteq U$. In the context of maximal consistent blocks, a fuzziness measure of the rough set X is defined as

$$E(F_X^A)=\sum_{i=1}^{|U|}\mu_X(u_i)(1-\mu_X(u_i)). \tag{27}$$

Theorem 20. *Let $S=(U,A)$ be an incomplete information system, in the context of maximal consistent blocks, the fuzziness measure of a crisp set equals zero.*

Proof. Let X be a crisp set in the incomplete information system $S=(U,A)$, then $\underline{apr}_A(X)=X=\overline{apr}_A(X)$. Hence, one can get that for every $u\in X$, $Y\subseteq X$ $(Y\in MC_A(u))$. In fact, if there exist some $Y\in MC_A(u)$ such that $Y\not\subseteq X$, i.e., $Y\cap X\neq\emptyset$, then $X\neq\overline{apr}_A(X)$. This yields a contradiction. Therefore, when $u\in X$, one has that $\mu_X(u)=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|X\cap Y|}{|Y|}=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|Y|}{|Y|}=1$. If $u\notin X$, we have that $\mu_X(u)=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|X\cap Y|}{|Y|}=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|\emptyset|}{|Y|}=0$. Thus, for any $u\in U$, $\mu_X(u_i)(1-\mu_X(u_i))=0$, i.e., $E(F_X^A)=0$. This completes the proof. □

Theorem 21. *Let $S=(U,A)$ be an incomplete information system, in the context of maximal consistent blocks, the fuzziness measure of a rough set is the same as that of its complement set.*

Proof. Let X be a rough set in the incomplete information system $S=(U,A)$ and X^c is its complement set on the universe U, i.e., $X^c=U-X$. For any $u\in U$, one has that

$$\begin{aligned}\mu_X(u)+\mu_{X^c}(u)&=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|X\cap Y|}{|Y|}+\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|X^c\cap Y|}{|Y|}\\&=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\left(\frac{|X\cap Y|}{|Y|}+\frac{|X^c\cap Y|}{|Y|}\right)\\&=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|Y|}{|Y|}\\&=1,\end{aligned}$$

i.e., $\mu_{X^c}(u)=1-\mu_X(u)$. Thus, for any $u\in U$, one can obtain that $\mu_X(u)(1-\mu_X(u))=\mu_{X^c}(u)(1-\mu_{X^c}(u))$, i.e., $E(F_X^A)=E(F_{X^c}^A)$. This completes the proof. □

Let $S=(U,A)$ be an incomplete information system and $U/D=\{D_1,D_2,\ldots,D_r\}$. For any object $u\in U$, in the context of maximal consistent blocks, the membership function of u in D is defined as

$$\mu_D(u)=\frac{1}{|MC_A(u)|}\sum_{Y\in MC_A(u)}\frac{|D_j\cap Y|}{|Y|}\quad (u\in D_j), \tag{28}$$

where $\mu_D(u)$ $(0 \leqslant \mu_D(u) \leqslant 1)$ represents a fuzzy concept. It can generate a fuzzy set $F_D^A = \{(u, \mu_X(u)) | u \in U\}$ on the universe U.

Definition 18. Let $S = (U, A)$ be an incomplete information system and $U/D = \{D_1, D_2, \ldots, D_r\}$. In the context of maximal consistent blocks, a fuzziness measure of a rough decision is defined as

$$E(F_D^A) = \sum_{i=1}^{|U|} \mu_D(u_i)(1 - \mu_D(u_i)), \tag{29}$$

where $\mu_D(u_i)$ denotes the membership function of $u_i \in U$ in the decision D.

In the following example, we show how to calculate the fuzziness of a rough decision in the context of maximal consistent blocks.

Example 8. (*continued from Example 7*). Suppose that $D_1 = \{u_1, u_2, u_4, u_6\}$, $D_2 = \{u_3\}$ and $D_3 = \{u_5\}$.

From formula (28), we have that

$\mu_D(u_1) = \frac{1}{|MC_A(u_1)|} \sum_{Y \in MC_A(u_1)} \frac{|D_j \cap Y|}{|Y|} = 1,$
$\mu_D(u_2) = \frac{1}{|MC_A(u_2)|} \sum_{Y \in MC_A(u_2)} \frac{|D_j \cap Y|}{|Y|} = 1,$
$\mu_D(u_3) = \frac{1}{|MC_A(u_3)|} \sum_{Y \in MC_A(u_3)} \frac{|D_j \cap Y|}{|Y|} = 1,$
$\mu_D(u_4) = \frac{1}{|MC_A(u_4)|} \sum_{Y \in MC_A(u_4)} \frac{|D_j \cap Y|}{|Y|} = \frac{1}{2},$
$\mu_D(u_5) = \frac{1}{|MC_A(u_5)|} \sum_{Y \in MC_A(u_5)} \frac{|D_j \cap Y|}{|Y|} = \frac{1}{2}(\frac{1}{2} + \frac{1}{2}) = \frac{1}{2}$ and
$\mu_D(u_6) = \frac{1}{|MC_A(u_6)|} \sum_{Y \in MC_A(u_6)} \frac{|D_j \cap Y|}{|Y|} = \frac{1}{2}(1 + \frac{1}{2}) = \frac{3}{4}.$

Therefore,

$$\begin{aligned} E(F_D^C) &= \sum_{i=1}^{6} \mu_D(u_i)(1 - \mu_D(u_i)) \\ &= 1 \times (1-1) + 1 \times (1-1) + 1 \times (1-1) + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{4} \\ &= \frac{11}{16}. \end{aligned}$$

Hence, the fuzziness measure of the rough decision induced by C in Table 1 in the context of maximal consistent blocks is $\frac{11}{16}$.

Theorem 22. *In an incomplete decision table $S = (U, C \cup D)$, in the context of maximal consistent blocks, the fuzziness measure of a crisp decision equals zero.*

Proof. Let $U/D = \{D_1, D_2, \ldots, D_r\}$ be a crisp decision in the context of maximal consistent block technique, i.e., $\underline{apr}_C(D_i) = D_i = \overline{apr}_C(D_i)$, $i = \{1, 2, \ldots, r\}$.

Hence, one has that for every $u \in D_i$, $Y \subseteq D_i$ $(Y \in MC_C(u))$. In fact, if there exist some $Y \in MC_A(u)$ such that $Y \not\subseteq X$, i.e., $Y \cap X \neq \emptyset$, then $X \neq \overline{apr}_A(X)$. This yields a contradiction. Therefore, when $u \in D_i$, it follows that

$$\mu_{D_i}(u) = \frac{1}{|MC_C(u)|} \sum_{Y \in MC_C(u)} \frac{|D_i \cap Y|}{|Y|} = \frac{1}{|MC_C(u)|} \sum_{Y \in MC_C(u)} \frac{|Y|}{|Y|} = 1;$$

when $u \notin D_i$, one can obtain that

$$\mu_{D_i}(u) = \frac{1}{|MC_C(u)|} \sum_{Y \in MC_C(u)} \frac{|D_i \cap Y|}{|Y|} = \frac{1}{|MC_C(u)|} \sum_{Y \in MC_C(u)} \frac{|\emptyset|}{|Y|} = 0.$$

Thus, for any $u \in U$, $\mu_{D_i}(u_i)(1 - \mu_{D_i}(u_i)) = 0$, i.e., $E(F_D^C) = 0$. This completes the proof. $\square$

Table 5
Values of consistency and fuzziness induced by each condition attribute in the context of maximal consistent blocks of Table 1

	Price	Mileage	Size	Max-Speed
Consistency	0.0833	0.1467	0.4000	0.3171
Fuzziness	1.3750	1.2800	0.8000	1.2135

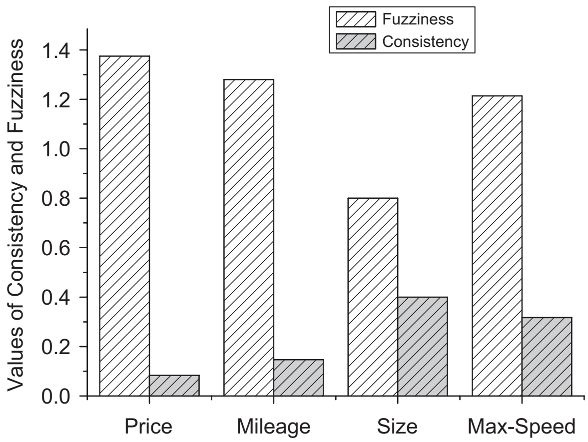

Fig. 3. Fuzziness and consistency induced by each condition attribute in the context of maximal consistent blocks of Table 1.

Finally, we show the validity of these two measures for constructing a heuristic function in the context of maximal consistent blocks in the incomplete decision table of Table 1. The values of the consistency and fuzziness induced by each condition attribute of Table 1 are shown in Table 5 and Fig. 3.

From Table 5 and Fig. 3, the two arrays of these four attributes can be obtained as follows.

(1) Consistency: Size→Max-Speed→Mileage→Price.
(2) Fuzziness: Size→Max-Speed→Mileage→Price.

The first array can be used to heuristically extract decision rules from an incomplete decision table in the context of maximal consistent blocks, the second array can be used to heuristically obtain the rough decision of a target decision in an incomplete decision table in the context of maximal consistent blocks. In the context of maximal consistent blocks, these two measures can also be regarded as the measures for evaluating the decision performance of an incomplete decision table.

6. Conclusions

Classical consistency degree and its two extensions can be used to measure the consistency of the condition part with respect to the decision part in three types of decision tables (complete, incomplete and maximal consistent blocks). However, they have some limitations when their values achieve zero. In this study, we have constructed the membership functions of an object through using the equivalence class, tolerance class and maximal consistent blocks obtaining itself, respectively. Based on these membership functions, we have introduced the consistency measures to assess the consistencies of a target concept and a decision table, and the fuzziness measures to compute the fuzziness of a rough set and a rough decision in these types of decision tables. In addition, we have established the relationships among the consistency, inclusion degree and fuzzy measure in three types of decision tables. Their validity have been shown by several illustrative examples and the experimental analyses on three kinds of decision tables. These results will be helpful for understanding the essence of uncertainty in decision tables and may be applied for rule extraction and rough classification in practical decision issues.

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