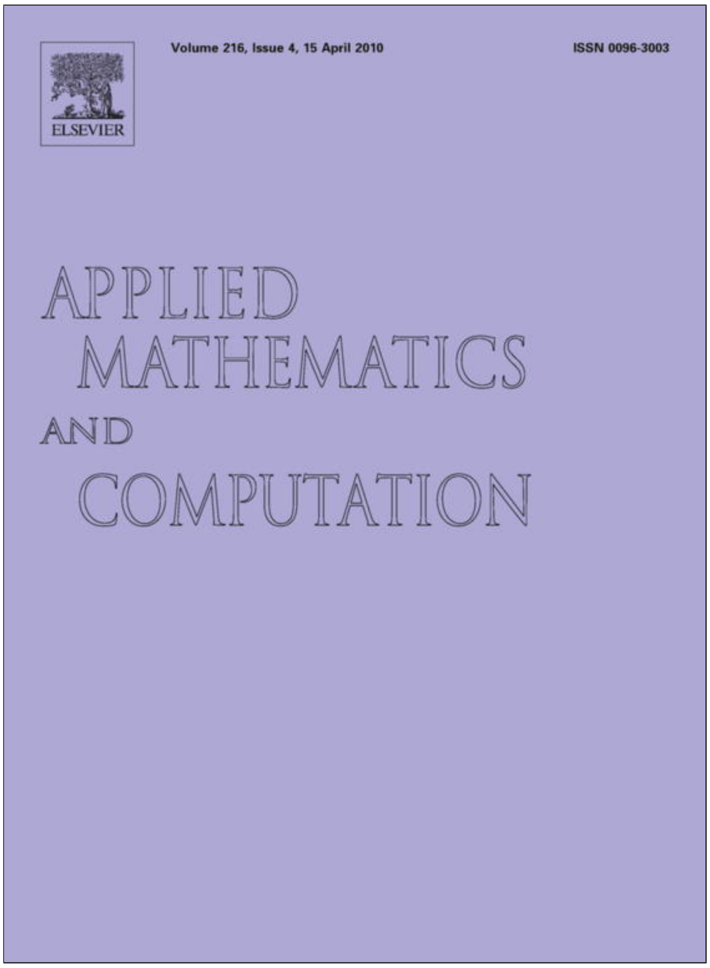
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Bifurcation and chaos in an epidemic model with nonlinear incidence rates

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ABSTRACT

This paper investigates a discrete-time epidemic model by qualitative analysis and numerical simulation. It is verified that there are phenomena of the transcritical bifurcation, flip bifurcation, Hopf bifurcation types and chaos. Also the largest Lyapunov exponents are numerically computed to confirm further the complexity of these dynamic behaviors. The obtained results show that discrete epidemic model can have rich dynamical behavior.

1. Introduction

Mathematical models describing the population dynamics of infectious diseases have been playing an important role in better understanding of epidemiological patterns and disease control for a long time. Epidemiological models are now widely used as more epidemiologists realize the role that modeling can play in basic understanding and policy development [10,19]. Understanding emergent infectious diseases in humans is viewed with increasing importance. The rapid spread of SARS [4], the perceived threat of bio-terrorism [15] and concerns over influenza pandemics [23] have all highlighted vulnerability to (re)emerging infections. For all these examples, mathematical modeling has been used to develop an understanding of the relevant epidemiology, as well as to quantify the likely effects of different intervention strategies [8,11,21].

An important aspect of the mathematical study of epidemiology is the formulation of the incidence function. The incidence rate is the rate of new infection. In most epidemiological models, bilinear and standard incidence rates have been frequently used in classical epidemic models [2,3,5,7,12–14,18]. Liu et al. [16,17] concluded that the bilinear mass action incidence rate due to saturation or multiple exposures before infection could lead to nonlinear incidence rate $\beta S^p I^q$.

Simple models, by their own nature, cannot incorporate many of the complex biological factors. However, they often provide useful insights to help our understanding of complex process. For such reason, in the present study, we set $p = 2$ and $q = 1$. We firstly focus on the following continuous model:

$$\begin{cases} \frac{dS}{d\tau} = rS(1 - \frac{S}{K}) - \beta S^2 I, \\ \frac{dI}{d\tau} = \beta S^2 I - dI, \end{cases} \tag{1.1}$$

where S, I are denoted as the susceptible and infected, respectively. And r represents the intrinsic birth rate constant, K represents the carrying capacity, β represents the force of infection or the rate of transmission, and d represents the death coefficient of I for the disease.

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By setting

$$\tau = \frac{t}{d}, \quad b = \frac{\beta}{d}, \quad a = \frac{r}{d},$$

we have the following form:

$$\begin{cases} \frac{dS}{dt} = aS(1 - \frac{S}{K}) - bS^2 I, \\ \frac{dI}{dt} = bS^2 I - I. \end{cases} \tag{1.2}$$

The advantages of a discrete-time approach are multiple in epidemic model [20,22]. Firstly, difference models are more realistic than differential ones since the epidemic statistics are compiled from given time intervals and not continuously. The second reason are that, the discrete-time models can provide natural simulators for the continuous cases. One can thus not only study with good accuracy the behavior of the continuous-time model, but also assess the effect of larger time steps. At last, the use of discrete-time models makes it possible to use the entire arsenal of methods recently developed for the study of mappings and lattice equations, either from the integrability and/or chaos points of view.

Applying Euler scheme to the system (1.2), we obtain the following equation:

$$\begin{cases} S_{n+1} = (a+1)S_n - cS_n^2 - bS_n^2 I_n, \\ I_{n+1} = bS_n^2 I_n, \end{cases} \tag{1.3}$$

where $c = \frac{a}{K}$.

The paper is organized as follows. It is verified that there are phenomena of the transcritical bifurcation, flip bifurcation and Hopf bifurcation in Section 2. In Section 3, a series of numerical simulations show that there are bifurcation and chaos in the discrete epidemic model. Finally, some conclusions are given.

2. Bifurcation analysis

For the Eq. (1.3), if the parameters a, b and c are fixed, by calculating, we can get the three fixed points $E_0 = (0,0)$, $E_1 = (\frac{a}{c}, 0)$ and $E_2 = \left(\frac{1}{\sqrt{b}}, \frac{a\sqrt{b}-c}{b}\right)$. It is obvious that the fixed point E_0 is a saddle. In the following sections, we will focus on E_1, E_2.

2.1. Fixed point $E_1 = (\frac{a}{c}, 0)$

The following is the Jacobian matrix at E_1:

$$J_{E_1} = \begin{pmatrix} 1-a & -\frac{ba^2}{c^2} \\ 0 & \frac{ba^2}{c^2} \end{pmatrix},$$

where a is a bifurcation parameter. If $ba^2 = c^2$, J_{E_1} has eigenvalues $\lambda_1 = 1 - a$, $\lambda_2 = 1$. And $a \neq 2$ implies $|\lambda_1| \neq 1$. The following theorem is the case that the fixed point E_1 is a transcritical bifurcation point.

Theorem 2.1. *If $ba^2 = c^2$, $a \neq 2$, the system (1.3) will undergoes a transcritical bifurcation at E_1. Moreover, when $b > \frac{c^2}{a^2}$, the system has three fixed points, and when $b \leqslant \frac{c^2}{a^2}$, the system has two fixed points.*

Proof. Let $\xi_n = S_n - \frac{a}{c}$, $\eta_n = I_n$, $\mu_n = b - \frac{c^2}{a_2}$, and parameter μ is a new and dependent variable, the system (1.3) becomes:

$$\begin{cases} \xi_{n+1} = (1-a)\xi_n - \eta_n - c\xi_n^2 - \frac{a^2}{c^2}\mu_n\eta_n - \frac{2c}{a}\xi_n\eta_n - \frac{c^2}{a^2}\xi_n^2\eta_n - \frac{2a}{c}\xi_n\mu_n\eta_n - \mu_n\xi_n^2\eta_n, \\ \eta_{n+1} = \eta_n + \frac{2c}{a}\xi_n\eta_n + \frac{a^2}{c^2}\mu_n\eta_n + \frac{c^2}{a^2}\xi_n^2\eta_n + \mu_n\xi_n^2\eta_n + \frac{2a}{c}\xi_n\mu_n\eta_n, \\ \mu_{n+1} = \mu_n. \end{cases} \tag{2.1}$$

Let

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the following transformation:

$$\begin{pmatrix} \xi_n \\ \eta_n \\ \mu_n \end{pmatrix} = T \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix},$$

then the system (2.1) can be changed into

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} 1-a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(u_n, v_n, \delta_n) \\ \tilde{f}_2(u_n, v_n, \delta_n) \\ 0 \end{pmatrix},$$

where

$$\tilde{f}_1(u_n, v_n, \delta_n) = \frac{c(a-2)v_n^2}{a} - cu_n^2 - \frac{2cu_n v_n}{a} + \frac{a^2(a-1)v_n\delta_n}{c^2} + \frac{2a(a-1)v_n\delta_n}{c}(u_n+v_n) + (a-1)\left(\frac{c^2}{a^2}+\delta_n\right)v_n(u_n+v_n)^2,$$

$$\tilde{f}_2(u_n, v_n, \delta_n) = \frac{a^2}{c^2}v_n\delta_n + \frac{2c}{a}(u_n+v_n)v_n + \frac{2a}{c}v_n\delta_n(u_n+v_n) + \left(\frac{c^2}{a^2}+\delta_n\right)(u_n+v_n)^2.$$

Then, we can consider

$$u_n = h(v_n, \delta_n) = a_1 v_n^2 + a_2 v_n\delta_n + a_3\delta_n^2 + o((|v_n| + |\delta_n|)^3),$$

which must satisfy:

$$\begin{aligned} h(v_n + \tilde{f}_2(h(v_n,\delta_n), v_n, \delta_n), \delta_{n+1}) &= (1-a)h(v_n,\delta_n) + \frac{c(a-2)v_n^2}{a} - ch^2(v_n,\delta_n) - \frac{2ch(v_n,\delta_n)v_n}{a} \\ &\quad + \frac{a^2(a-1)v_n\delta_n}{c^2} + \frac{2a(a-1)v_n\delta_n}{c}(h(v_n,\delta_n)+v_n) \\ &\quad + (a-1)\left(\frac{c^2}{a^2}+\delta_n\right)v_n(h(v_n,\delta_n)+v_n)^2. \end{aligned}$$

Thus, we can get that

$$a_1 = \frac{c(a-2)}{a^2}, \quad a_2 = \frac{a(a-1)}{c^2}, \quad a_3 = 0.$$

And the system (2.1) is restricted to the center manifold, which is given by:

$$f_2 : v_{n+1} = v_n + \frac{2c}{a}v_n^2 + \frac{a^2}{c^2}v_n\delta_n + \frac{c^2(3a-4)}{a^3}v_n^3 + \frac{2(2a-1)}{c}v_n^2\delta_n + o((|v_n| + |\delta_n|)^4).$$

Since $f_2(0,\delta_n) = 0$, $\frac{\partial f_2}{\partial v}|_{(0,0)} = 1$, $\frac{\partial^2 f_2}{\partial v^2}|_{(0,0)} = \frac{4c}{a} \neq 0$, $\frac{\partial^2 f_2}{\partial v \partial \delta}\Big|_{(0,0)} = \frac{a^2}{c^2} \neq 0$, system (1.3) undergoes a transcritical bifurcation at E_1. The proof is completed. □

Theorem 2.2. *If $a = 2$, $4b \neq c^2$, the system (1.3) will undergoes a flip bifurcation at E_1. Moreover, the stable periodic-2 points bifurcate from this fixed point.*

Proof. Let $\xi_n = S_n - \frac{a}{c}$, $\eta_n = I_n$, $\mu_n = a - 2$, and parameter μ is a new and dependent variable, the system (1.3) becomes:

$$\begin{cases} \xi_{n+1} = -\xi_n - \frac{4b}{c^2}\eta_n - c\xi_n^2 - \xi_n\mu_n - \frac{4b}{c^2}\mu_n\eta_n - \frac{4b}{c}\xi_n\eta_n - b\xi_n^2\eta_n - \frac{2b}{c}\xi_n\mu_n\eta_n - \frac{b}{c^2}\mu_n^2\eta_n, \\ \mu_{n+1} = -\mu_n, \\ \eta_{n+1} = \frac{4b}{c^2}\eta_n + \frac{4b}{c}\xi_n\eta_n + \frac{4b}{c^2}\mu_n\eta_n + b\xi_n^2\eta_n + \frac{2b}{c}\xi_n\mu_n\eta_n + \frac{b}{c^2}\mu_n^2\eta_n. \end{cases} \tag{2.2}$$

Let

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{4b+c^2}{4b} \end{pmatrix}.$$

By the following transformation:

$$\begin{pmatrix} \xi_n \\ \mu_n \\ \eta_n \end{pmatrix} = T \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix},$$

then the system (2.2) can be changed into

$$\begin{pmatrix} u_{n+1} \\ \delta_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{4b}{c^2} \end{pmatrix} \begin{pmatrix} u_n \\ \delta_n \\ v_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(u_n, v_n, \delta_n) \\ 0 \\ \tilde{f}_2(u_n, v_n, \delta_n) \end{pmatrix},$$

where

$$\tilde{f}_1(u_n, v_n, \delta_n) = -cu_n^2 - cu_n v_n - u_n\delta_n + \frac{c^2}{4}v_n^3 + \frac{1}{4}v_n\delta_n^2 + \frac{c^2}{2}u_n v_n\left(\frac{1}{2}u_n + v_n\right) + \frac{c}{2}\delta_n v_n(u_n + v_n),$$

$$\tilde{f}_2(u_n, v_n, \delta_n) = \frac{4b}{c^2}v_n\delta_n + \frac{4b}{c}v_n(u_n + v_n) + \frac{2b}{c}v_n\delta_n(u_n + v_n) + bv_n(u_n + v_n)^2 + \frac{b}{c^2}v_n\delta_n^2.$$

Then, we can consider

$$v_n = h(u_n, \delta_n) = b_1u_n^2 + b_2u_n\delta_n + b_3\delta_n^2 + o((|u_n| + |\delta_n|)^3),$$

which must satisfy:

$$h(-u_n + \tilde{f}_1(u_n, h(u_n, \delta_n), \delta_n), \delta_{n+1}) = \frac{4b}{c^2}h(u_n, \delta_n) + \frac{4b}{c^2}h(u_n, \delta_n)\delta_n + \frac{4b}{c}h(u_n, \delta_n)(u_n + h(u_n, \delta_n)) + \frac{2b}{c}h(u_n, \delta_n)\delta_n(u_n + h(u_n, \delta_n)) + \frac{b}{c^2}h(u_n, \delta_n)\delta_n^2 + bh(u_n, \delta_n)(u_n + h(u_n, \delta_n))^2.$$

By calculating, we can get that $b_1 = b_2 = b_3 = 0$. We take the center manifold as the following form:

$$v_n = h(u_n, \delta_n) = b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3 + o((|u_n| + |\delta_n|)^4),$$

and the system (2.2) is restricted to the center manifold, which is given by:

$$\begin{aligned} f_1 : u_{n+1} = {} & -u_n - cu_n^2 - cu_n(b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3) - u_n\delta_n \\ & + \frac{c^2}{4}(b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)^3 + \frac{1}{4}(b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)\delta_n^2 \\ & + \frac{c^2}{2}u_n(b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)\left[\frac{1}{2}u_n + (b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)\right] \\ & + \frac{c}{2}\delta_n(b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)[u_n + (b_4u_n^3 + b_5u_n^2\delta_n + b_6u_n\delta_n^2 + b_7\delta_n^3)]. \end{aligned}$$

Since

$$\left.\left(\frac{\partial f_1}{\partial \delta}\frac{\partial^2 f_1}{\partial u^2} + 2\frac{\partial^2 f_1}{\partial u\partial \delta}\right)\right|_{(0,0)} = -2 \neq 0,$$

and

$$\left.\left(\frac{1}{2}\left(\frac{\partial^2 f_1}{\partial u^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f_1}{\partial u^3}\right)\right)\right|_{(0,0)} = 2c^2 > 0,$$

system (1.3) undergoes a flip bifurcation at E_1. The proof is completed. □

2.2. Fixed point $E_2 = \left(\frac{1}{\sqrt{b}}, \frac{a\sqrt{b}-c}{b}\right)$

In this section, we will pay our attention to the fixed point $E_2\left(\frac{1}{\sqrt{b}}, \frac{a\sqrt{b}-c}{b}\right)$, which is the only positive fixed point of the system (1.3). The positive fixed point is so important to the biological system that people usually are very interested in it. In the following two theorems it will be showed that the system (1.3) also undergoes transcritical and flip bifurcation at E_2. Since the analysis is very similar to the case at E_1, the proof are omitted.

Theorem 2.3. *If $a^2 - 8a + \frac{8c}{b}\sqrt{b} > 0, ab = c\sqrt{b}$, the system (1.3) will undergoes a transcritical bifurcation at E_2.*

Theorem 2.4. *If $a^2 - 8a + \frac{8c}{b}\sqrt{b} > 0, 2b = c\sqrt{b}$, the system (1.3) will undergoes a flip bifurcation at E_2.*

Next, the Hopf bifurcation at E_2 will be discussed.

Theorem 2.5. *If $a^2 - 8a + \frac{8c}{b}\sqrt{b} < 0, a_0 = \frac{2c}{\sqrt{b}}$, and $a \neq 2$, the system (1.3) will undergoes a Hopf bifurcation at E_2. Moreover, for $a > \frac{2c}{\sqrt{b}}$ an attracting invariant closed curve bifurcates from the fixed point.*

Proof. Let

$$\xi_n = S_n - \frac{1}{\sqrt{b}}, \quad \eta_n = I_n - \frac{a\sqrt{b} - c}{b}.$$

Then the system (1.3) can be changed into:

$$\begin{cases} \xi_{n+1} = (1-a)\xi_n - \eta_n - a\sqrt{b}\xi_n^2 - 2\sqrt{b}\xi_n\eta_n - b\xi_n^2\eta_n, \\ \eta_{n+1} = \left(2a - \frac{2c}{\sqrt{b}}\right)\xi_n + \eta_n + \left(a\sqrt{b} - c\right)\xi_n^2 + 2\sqrt{b}\xi_n\eta_n + b\xi_n^2\eta_n. \end{cases} \tag{2.3}$$

Now take a as an bifurcation parameter. For $8a - a^2 - \frac{8c}{b}\sqrt{b} > 0$, the eigenvalues of the J_{E_2} are

$$\lambda_{1,2} = \frac{2 - a \pm i\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}}}{2}$$

and

$$|\lambda| = \frac{1}{2}\sqrt{(2-a)^2 + 8a - a^2 - \frac{8c}{b}\sqrt{b}} = \sqrt{a + 1 - \frac{2c}{\sqrt{b}}}.$$

Let $a_0 = \frac{2c}{\sqrt{b}}$, then

$$\left.\frac{d(|\lambda|)}{da}\right|_{a_0 = \frac{2c}{\sqrt{b}}} = \frac{1}{2\sqrt{a + 1 - \frac{2c}{\sqrt{b}}}} = \frac{1}{2} > 0,$$

$|\lambda(a_0)| = 1$, and

$$\lambda_{1,2}(a_0) = \left.\frac{2 - a \pm i\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}}}{2}\right|_{a_0 = \frac{2c}{\sqrt{b}}} = 1 - \frac{c}{\sqrt{b}} \pm i\sqrt{\frac{2c}{\sqrt{b}} - \frac{c^2}{b}}.$$

If $\frac{c}{\sqrt{b}} \neq 1,\ \lambda_{1,2}^m \neq 1,\ m = 1,2,3,4$. By the following transformation:

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = T\begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

where

$$T = \begin{pmatrix} 1 & 0 \\ -\frac{a}{2} & \frac{1}{2}\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}} \end{pmatrix}.$$

The system (2.3) can be changed into:

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(2-a) & -\frac{1}{2}\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}} \\ \frac{1}{2}\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}} & \frac{1}{2}(2-a) \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(u_n, v_n) \\ \tilde{f}_2(u_n, v_n) \end{pmatrix},$$

where

$$\tilde{f}_1(u_n, v_n) = \sqrt{8ab - a^2b - 8c\sqrt{b}}u_n v_n + \frac{ab}{2}u_n^3 - \frac{b}{2}\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}}u_n^2 v_n,$$

and

$$\tilde{f}_2(u_n, v_n) = -cu_n^2 + \sqrt{8ab - a^2b - 8c\sqrt{b}}u_n v_n - \frac{ab}{2}u_n^3 + \frac{b}{2}\sqrt{8a - a^2 - \frac{8c}{b}\sqrt{b}}u_n^2 v_n.$$

In [21], the coefficient θ is given by

$$\theta = -Re\left[\frac{(1-2\lambda)\bar{\lambda}^2}{1-\lambda}l_{11}l_{20}\right] - \frac{1}{2}|l_{11}|^2 - |l_{02}|^2 + Re(\bar{\lambda}l_{21}),$$

where

$$l_{20} = \frac{1}{8}\left[\left(\tilde{f}_{1uu} - \tilde{f}_{1vv} + 2\tilde{f}_{2uv}\right) + i\left(\tilde{f}_{2uu} - \tilde{f}_{2vv} - 2\tilde{f}_{1uv}\right)\right] = \frac{\sqrt{8ab - a^2b - 8c\sqrt{b}}}{4} + \frac{1}{4}i\left(-c - \sqrt{8ab - a^2b - 8c\sqrt{b}}\right),$$

$$l_{11} = \frac{1}{4}\left[\left(\tilde{f}_{1uu} + \tilde{f}_{1vv}\right) + i\left(\tilde{f}_{2uu} + \tilde{f}_{2vv}\right)\right] = -\frac{c}{2}i,$$

$$l_{02} = \frac{1}{8}\left[\left(\tilde{f}_{1uu} - \tilde{f}_{1vv} - 2\tilde{f}_{2uv}\right) + i\left(\tilde{f}_{2uu} - \tilde{f}_{2vv} + 2\tilde{f}_{1uv}\right)\right] = -\frac{\sqrt{8ab - a^2b - 8c\sqrt{b}}}{4} + \frac{1}{4}i\left(-c + \sqrt{8ab - a^2b - 8c\sqrt{b}}\right),$$

$$l_{21} = \frac{1}{16}\left[\left(\tilde{f}_{1uuu} + \tilde{f}_{1uvv} + \tilde{f}_{2uuv} + \tilde{f}_{2vvv}\right) + i\left(\tilde{f}_{2uuu} + \tilde{f}_{2uvv} - \tilde{f}_{1uuv} - \tilde{f}_{1vvv}\right)\right]$$
$$= \frac{3ab}{16} + \frac{b\sqrt{8a - a^2 - \frac{8c}{\sqrt{b}}}}{16} + \frac{1}{16}i\left(-3ab + b\sqrt{8a - a^2 - \frac{8c}{\sqrt{b}}}\right).$$

By direct calculating, we obtain that $\theta < 0$. Using the Hopf bifurcation theorem in [6], the proof is completed. □

3. Numerical simulations

To provide some numerical evidence for the qualitative dynamic behavior of the model (1.3), the phase portraits, bifurcation diagrams, Lyapunov exponents, sensitive dependence on initial conditions and fractal dimension were used to illustrate the above analytical results and for finding new dynamics as the parameters vary.

Now, a is considered as a parameter with the range (0,1). A powerful numerical tool to investigate whether the dynamical behavior is chaotic is a plot of the largest Lyapunov exponent, as a function of one of the model parameters. The largest Lyapunov exponent is the average growth rate of an infinitesimal state perturbation along a typical trajectory (orbit). Fig. 1a shows the spectrum of Lyapunov exponents of the system (1.3) with respect to parameter a, and the parameter values are that $b = 5$ and $c = 0.5$. Since that the bifurcation diagrams of $a - S_n$ is similar with the bifurcation diagrams of $a - I_n$, we will only show the former which can be seen from Fig. 1b. From Fig. 1b, we can see that period-4 occurs at $a \approx 0.19354839$, period-5 occurs at $a \approx 0.35483871$ and period-6 occurs at $a \approx 0.55268817$. Cascades of period-halving bifurcations and period-doubling bifurcations can be sen from Fig. 1b. And as a increases from 0.50322581 to 0.55653763, the system goes through quasi-periodicity, including frequency-lockings and tangent bifurcation. When a is increased to $a_c(a_c \approx 0.86236556)$, system (1.3) becomes stable. To well see the dynamics, time series of S_n given in Fig. 2.

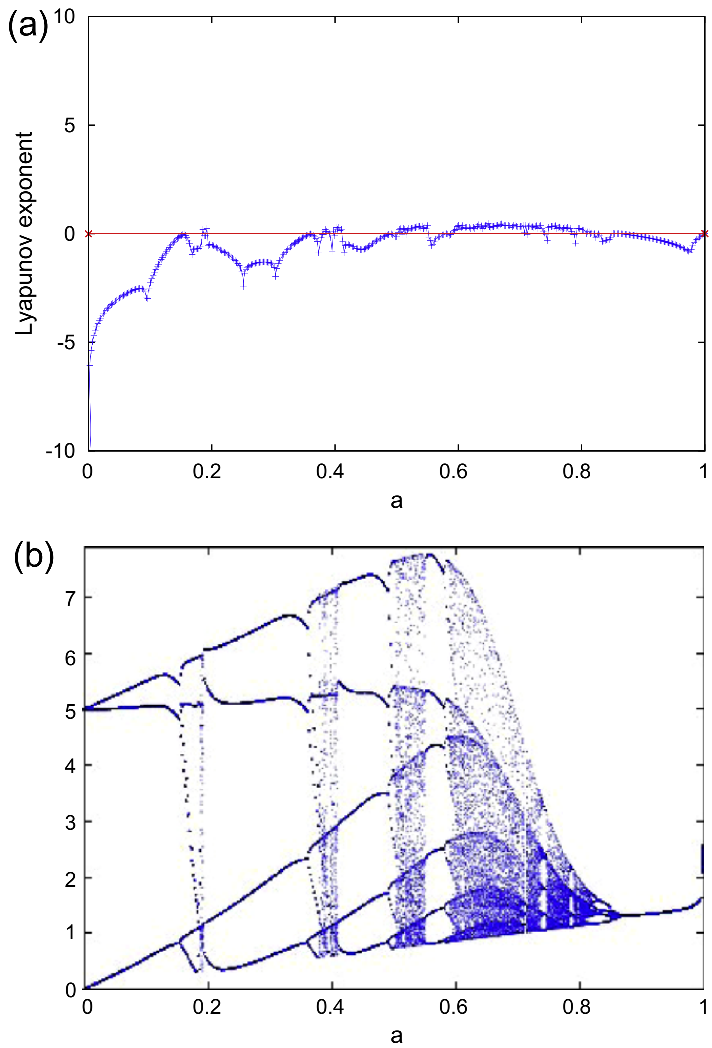

Fig. 1. Spectrum of Lyapunov exponents and bifurcation diagrams for a—S_n with $b = 5$ and $c = 0.5$.

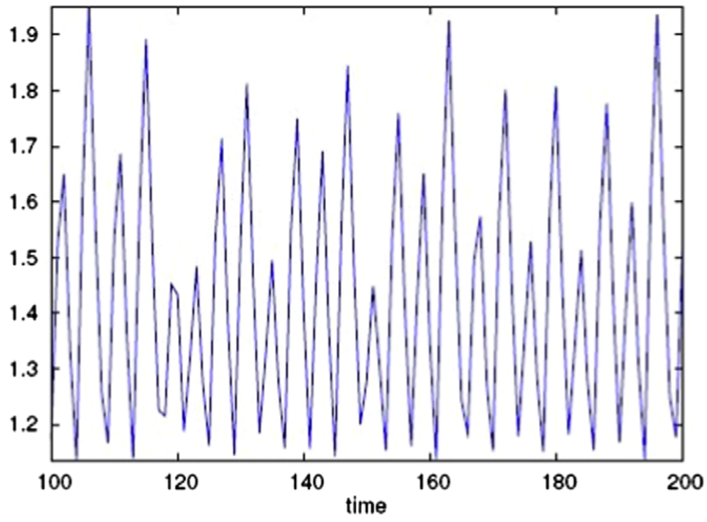

Fig. 2. Time series of S_n with $a = 0.3$, $b = 5$ and $c = 0.5$.

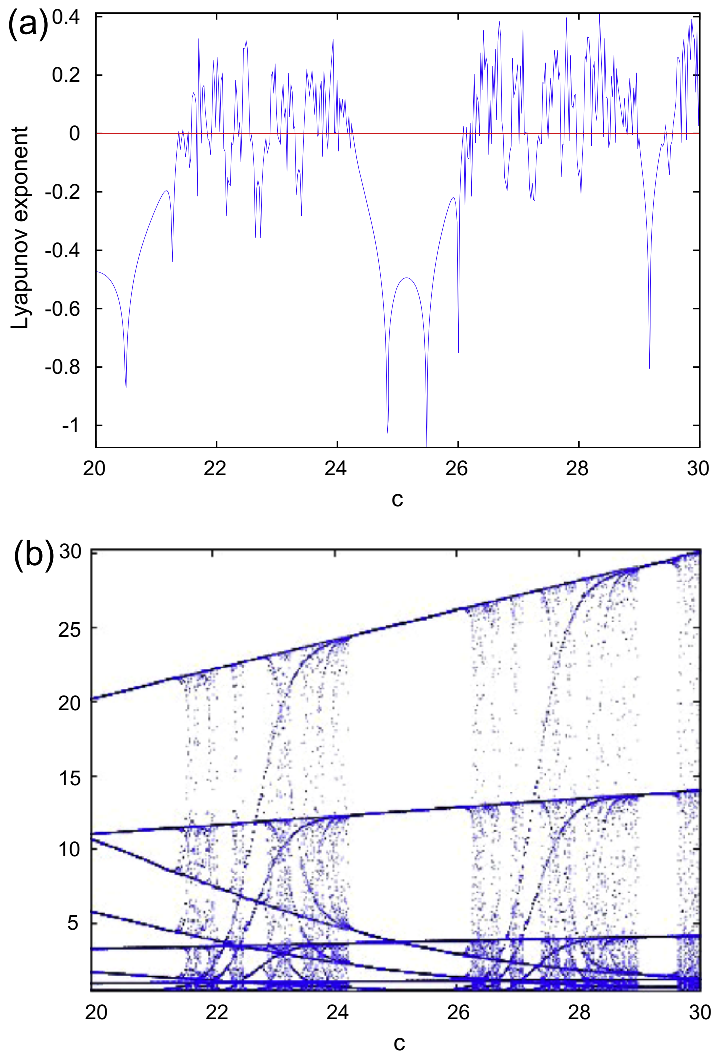

Fig. 3. Spectrum of Lyapunov exponents and bifurcation diagrams for c—S_n with $a = 0.3$ and $b = 5$.

Since c may play a important role of the system, we take it as a parameter. Fig. 3a shows the spectrum of Lyapunov exponents of the system (1.3) with respect to parameter c, and the parameter values are that $a = 0.3$, and $b = 5$. Fig. 3b is the bifurcation diagram of system (1.3) for the S_n. From Fig. 4, we can see that there is period-6 solution.

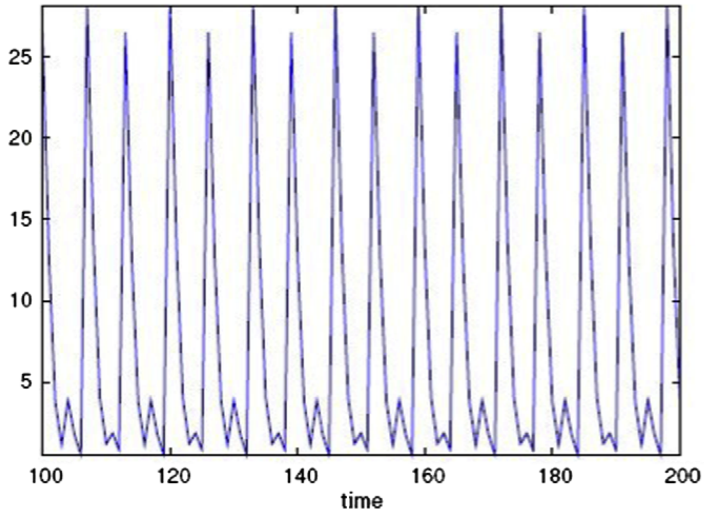

Fig. 4. Time series of S_n with $a = 0.3$, $b = 5$, and $c = 28$.

4. Discussion and conclusion

In this paper, we have applied Euler scheme to convert the continuous epidemic model to a discrete model and studied the dynamical characteristic of the discrete model. The discrete model can result in a much richer set of patterns than the corresponding continuous-time model and it is more effective in practice. Our theoretical analysis and numerical simulations have demonstrated that the discrete epidemic model undergoes transcritical bifurcation, flip bifurcation, Hopf bifurcation and chaos.

Furthermore, chaos can cause the population to run a higher risk of extinction due to the unpredictability [1,9]. Also the density of the infected may be out of control. But in the real world, the density of the infected needs to be under control or it will be harmful to the health of people worldwide. Thus, how to control chaos in the epidemic model is very important, which needs further investigation.

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