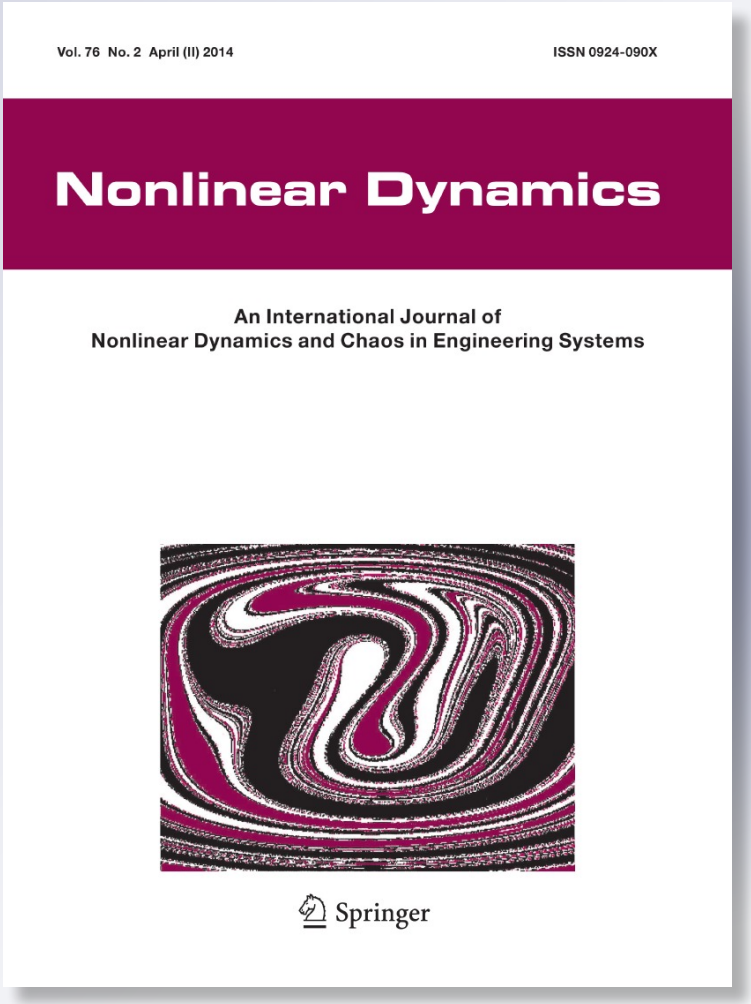
Periodic solutions of an epidemic model with saturated treatment

Li Li, Yanping Bai & Zhen Jin

Nonlinear Dynamics
An International Journal of Nonlinear Dynamics and Chaos in Engineering Systems

ISSN 0924-090X
Volume 76
Number 2

Nonlinear Dyn (2014) 76:1099-1108
DOI 10.1007/s11071-013-1193-0



Springer

Nonlinear Dyn (2014) 76:1099–1108
DOI 10.1007/s11071-013-1193-0

ORIGINAL PAPER

Periodic solutions of an epidemic model with saturated treatment

Li Li · Yanping Bai · Zhen Jin

Received: 7 September 2012 / Accepted: 13 December 2013 / Published online: 25 December 2013

Abstract Based on the fact that many infectious diseases exhibit periodic fluctuations and there is a saturated phenomenon during disease treatment, we study an SIR model with periodic incidence rate and saturated treatment function. Firstly, we find that the basic reproduction number less than 1 cannot insure the global stability of disease-free equilibrium and it needs to add other conditions. Moreover, we establish sufficient conditions for the multiplicity of positive periodic solutions. We also apply the numerical method to confirm theoretical results and show the stability of the periodic solutions. We observe that there are two periodic solutions in the system where one is stable and the other one is unstable. These results will provide some guidance for control measures of disease.

L. Li (✉)
School of Information and Communication Engineering, North University of China, Taiyuan 030051, Shan'xi, People's Republic of China
e-mail: lili831113@163.com

L. Li
Department of Mathematics, Taiyuan Institute of Technology, Taiyuan 030008, Shan'xi, People's Republic of China

Y. Bai
Department of Mathematics, North University of China, Taiyuan 030051, Shan'xi, People's Republic of China

Z. Jin
Complex Systems Research Center, Shanxi University, Taiyuan 030006, Shan'xi, People's Republic of China
e-mail: jinzhn@263.net

Keywords SIR model · Saturated treatment · Basic reproduction number · Periodic solutions

1 Introduction

It can be observed that many infectious diseases exhibit periodic fluctuations, such as pertussis (whooping cough), measles (rubeola), influenza, polio, chickenpox, mumps, rabies, etc. Specifically, pertussis is a contagious disease which can only spread from person to person. Through close contact with others, people who develop pertussis usually cough or sneeze to spread the disease. Pertussis most commonly affects infants and young children, especially babies less than one year of age. Someone with pertussis often needs to take deep breaths which lead to a "whooping" sound. Pertussis has been an endemic (common) disease in the United States, with a 3–5-year cycle and frequent outbreaks [1]. Measles is also a highly contagious viral disease which affects mostly children and can be transmitted via droplets from the nose, mouth, or throat of infected persons. There is no specific treatment for measles and most people recover within 2–3 weeks. However, it can be prevented by immunization [2]. Measles occurs in a seasonal pattern, which may still occur every 2 or 3 years in areas where there is low vaccine coverage. Its spread can be strengthened during the late winter and early spring in temperate climates and after the rainy season in tropical climates [3]. Influenza viruses which are familiar to all of us circulate

in animals and pose threats to human health. Typically, human cases can be attributed to viruses from animal sources, such as avian influenza virus subtypes H5N1 and H9N2 and swine influenza virus subtypes H1N1 and H3N2. The primary risk factor for human infection appears to be direct or indirect exposure to infected live or dead animals or contaminated environments [4]. We know that influenza has seasonality and repeatability. As mentioned in the WHO Director-General's post-pandemic announcement in August 2010, the pandemic A(H1N1) 2009 virus was expected to continue to circulate and cause local outbreaks and epidemics [5]. Rabies is a preventable viral disease of mammals, most often transmitted through the bite of a rabid animal. The rabies virus infects the central nervous system, ultimately causing disease in the brain and death [6]. Once these symptoms develop, the mortality is almost 100 %. Moreover, the monthly data of human rabies cases reported by the Chinese Ministry of Health also exhibit a periodic pattern on an annual base.

Seasonal variations can exert strong pressures on population dynamics. In addition to driving temporal patterns, from an applied perspective, exposing the mechanisms that link seasonal environmental changes to disease dynamics will aid in forecasting long-term health risks and in proposing control strategies. There appear to be several reasons for this. First, the cause of seasonality partly is climate, such as weather, humidity, and so on, which can influence the survivance of virus or bacteria [7–10]. Second, human or animals activity can be relevant to the incidence of disease [11,12]. Moveover, their relative importance depends on the local context. For example, winter peaks in the incidence of measles in temperate regions are likely to be caused not only by school terms but also by lower indoor humidity, which favors survival of the virus in the air [13]. In the tropics, measles incidence peaks during the dry season and the association with school terms are not apparent. Instead, increased survival of the virus in the dryer air may be the key determinant of these dry season peaks [14]. However, the mechanisms responsible for seasonal disease incidence and the epidemiological consequences of seasonality are poorly understood with rare exception. So, investigating how a periodic phenomenon arises is a hotspot for many researchers. First, the differential equations with delay or nonlinear incidence rate can have periodic solutions [15–18]. Moreover, seasonality can cause population fluctuations ranging from annual cycles to multiyear oscillations, and even chaotic dynamics [19–21]. In general, seasonally effective contact rate [22–28], and periodic changing in the birth rate [29] and vaccination program are often regarded as sources of periodicity [30].

With regard to mathematical describing of periodicity, seasonal transmission is often assumed to be sinusoidal, such that $\lambda(t) = \lambda(1 + \eta \sin(\frac{\pi t}{b} + c))$ where η is the amplitude of seasonal variation in transmission (typically referred to as the "strength of seasonal forcing"), $2b$ is the period, and c is phase difference, which is a crude assumption for many infectious diseases [14,31,32]. When $\eta = 0$, there are no nonseasonal infections. Sometimes, cosine function is also taken to describe the fluctuation. Motivated by biological realism, some recent papers take the contact rate as $\lambda(t) = \lambda(1 + \eta \mathrm{term(t)})$, where term is a periodic function which is +1 during the school term and -1 during school holidays. A more natural term can be written as $\lambda(t) = \lambda(1 + \eta)^{\mathrm{term(t)}}$ [31]. In this paper, we will take the form $\lambda(t) = a + \eta \sin \frac{\pi t}{6}$ which is a special circumstance.

Besides the periodic transmission rate, we adopt a saturated incidence rate and a saturated treatment rate. When the scale of the population is relatively small, the bilinear incidence rate is reasonable. Moreover, when it is relatively large, the standard incidence rate is good. However, they are both extreme cases. So, in 1978, Capasso and Serio [33] introduced a saturated incidence rate $g(I)S$ into the epidemic model, where $g(I) = \frac{kI}{1+\alpha I}$. kI measures the infection force of the disease and $\frac{kI}{1+\alpha I}$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases. When I is small, $g(I)$ and I are in direct proportion approximately. $g(I)$ tends toward a saturation level when I gets large. So, the saturated incidence rate is more reasonable and it has been investigated by many papers [34–36].

The saturated treatment rate in our paper is used to describe the effect of the infected individuals being delayed for treatment due to the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals [37]. At the beginning of the outbreak, owing to shortage of effective treatment techniques, the treatment rate is smaller. Then, as there is improvement of the hospital treatment conditions including effective medicines, skillful techniques, better understanding, etc., the treatment rate will be increased. At last, because the treatment capacity of

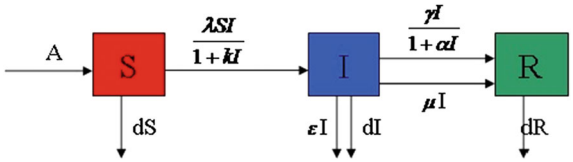


Fig. 1 The transmission diagram

any community is limited, the treatment will reach its saturation when the number of infective individuals is large enough [38]. In view of the above description, we suppose the treatment term to be $\frac{\gamma I}{1+\alpha I}$, where γ and α are undetermined parameters. Thus, the purpose of this paper is to investigate an SIR model with saturated and periodic incidence rate and saturated treatment function based on [37], to study its dynamic behavior and to establish sufficient conditions for the multiplicity of positive periodic solutions.

The article is organized as follows. In Sect. 2, we introduce the model, study the global asymptotic stability of the disease-free equilibrium and the existence of positive periodic solutions, and give sufficient conditions for the existence of two positive periodic solutions. Simulations of stability of positive periodic solutions are performed in Sect. 3. In Sect. 4, we give a brief discussion.

2 Mathematical modeling and analysis

2.1 Model formulation

We consider an SIR model with saturated and periodic incidence rate and saturated treatment function, whose corresponding autonomous model has been studied in [37]. The population is divided into three classes: the susceptible class denoted by S, the infectious class denoted by I, and the recovered class denoted by R. The transition dynamics associated with these subpopulations are illustrated in Fig. 1.

The model is a system of ordinary differential equations:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - \mathrm{d}S - \frac{\lambda(t)SI}{1+kI}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\lambda(t)SI}{1+kI} - (d+\varepsilon+\mu)I - \frac{\gamma I}{1+\alpha I}, \\ \frac{\mathrm{d}R}{\mathrm{d}t} = \mu I + \frac{\gamma I}{1+\alpha I} - \mathrm{d}R. \end{cases} \tag{1}$$

where $\lambda(t) = a + \eta \sin \frac{\pi}{6} t$ and other parameters are positive. The interpretation and values of parameters are described in Table 1.

Table 1 Descriptions and values of parameters in model (1)

Parameter	Interpretation
A	The recruitment rate of the population
k	The auxiliary parameter
d	The natural mortality rate
a	The baseline contact rate
η	The magnitude of forcing
μ	The natural recovery rate of the infective
ϵ	The disease-related death rate
γ	The auxiliary parameter
α	The auxiliary parameter

Noticing the equations in model (1), we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - \mathrm{d}N - \epsilon I. \tag{2}$$

Let $X = \{(S, I, R) | S, I, R \geq 0, 0 < S+I+R \leq \frac{A}{d}\}$.

Theorem 2.1 *The region X is positively invariant with respect to system (1).*

Because the third equation is independent of the first two equations in system (1), we only need to study the following reduced system,

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - \mathrm{d}S - \frac{\lambda(t)SI}{1+kI}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\lambda(t)SI}{1+kI} - (d+\varepsilon+\mu)I - \frac{\gamma I}{1+\alpha I}. \end{cases} \tag{3}$$

2.2 Global stability of the disease-free equilibrium

It is easy to see that system (1) has one disease-free equilibrium

$$E_0 = (S_0, 0, 0),$$

where $S_0 = A/d$. We can evaluate the basic reproduction number R_0 for system (1) following the definition of [39–43]

$$R_0 = \frac{\bar{\lambda} S_0}{d+\epsilon+\mu+\gamma},$$

where $\bar{\lambda} = \frac{1}{\omega}\int_0^\omega \lambda(t)\mathrm{d}t$ and ω is the period.

Theorem 2.2 *The disease-free equilibrium E_0 is globally, asymptotically stable when (1) $R_0 < 1$ and $k > \alpha$. Or (2) $R_0^c = \frac{\bar{\lambda} S_0}{d+\epsilon+\mu} < 1$.*

Proof (1) By [44], we know that E_0 is locally stable, if $R_0 < 1$. So, we only need to prove that E_0 is globally attractive for $R_0 < 1$. We know that when $R_0 < 1$, $R_0 + \frac{\bar{\lambda}\eta}{d+\epsilon+\mu+\gamma} < 1$, that is, $\bar{\lambda}(S_0+\eta) < d+\epsilon+\mu+\gamma$ for a small enough positive number η. Assume that $(S(t), I(t))$ is a nonnegative solution of system (2). Then, from the first equation in system (2), we can obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t} = A - \mathrm{d}S - \frac{\lambda(t)SI}{1+kI} \le A - \mathrm{d}S.$$

For the comparison system $\frac{\mathrm{d}S'}{\mathrm{d}t} = A - \mathrm{d}S'$, there exists a unique and global, asymptotically stable equilibrium $S_0 = \frac{A}{d}$. Thus, by the standard comparison theorem in [45], it can be obtained that for any small enough $\eta > 0$, there exists a t_0 such that $S(t) < S_0 + \eta$ for all $t > t_0$, which can be followed by

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\lambda(t)SI}{1+kI} - (d+\varepsilon+\mu)I - \frac{\gamma I}{1+\alpha I}\\ &\le \left[\bar{\lambda}(S_0+\eta) - (d+\varepsilon+\mu) - \frac{\gamma}{1+\alpha I}\right] I\\ &< \left(\frac{d+\varepsilon+\mu+\gamma}{1+kI} - (d+\varepsilon+\mu) - \frac{\gamma}{1+\alpha I}\right) I\\ &\le \left(\frac{\gamma}{1+kI} - \frac{\gamma}{1+\alpha I}\right) I.\end{aligned}$$

When $k > \alpha$, $\frac{\mathrm{d}I}{\mathrm{d}t} < 0$. Thus, for $t \to +\infty$, $I \to 0$, $S \to \frac{A}{d}$. So, the disease-free equilibrium E_0 is globally attractive, that is, it is global asymptotically stable.

(2) If $R_0^c = \frac{\bar{\lambda}S_0}{d+\epsilon+\mu} < 1$, $R_0 < 1$. So, we only need to prove E_0 is globally attractive. We also know that when $R_0^c < 1$, $R_0^c + \frac{\bar{\lambda}\eta}{d+\epsilon+\mu} < 1$ for a small enough number η. Following (1), for any small enough $\eta > 0$, there exists a t_0 such that $S(t) < S_0 + \eta$ for all $t > t_0$, which can be followed by

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\lambda(t)SI}{1+kI} - (d+\varepsilon+\mu)I - \frac{\gamma I}{1+\alpha I}\\ &\le [\bar{\lambda}(S_* + \eta) - (d+\varepsilon+\mu)]I\\ &< (d+\varepsilon+\mu)(R_0^c - 1)I\\ &< 0.\end{aligned}$$

Similar to (1), the disease-free equilibrium E_0 is globally attractive, that is, it is global asymptotically stable. □

The proof is completed.

2.3 Existence of positive periodic solutions

Define

$$X = \mathbb{R}^2_+,\ X_0 := \{(S, I) \in X : I > 0\},\ \partial X_0 = X \backslash X_0$$

and denote $u(t, x_0)$ as the unique solution of system (2) with the initial value $x_0 = (S^0, I^0)$. Let $P : X \to X$ be the Poincaré map associated with system (3), i.e.,

$$P(x_0) = u(\omega, x_0),\ \forall x_0 \in X,$$

where ω is the period. Applying the fundamental existence uniqueness theorem [46], we know that $u(t, x_0)$ is the unique solution of system (1) with $u(0, x_0) = x_0$. From Theorem 2.1, we know that X is positively invariant and P is point dissipative.

Lemma 2.3 *When $R_0 > 1$, there exists a constant $\delta > 0$ such that when*

$$\|(S^0, I^0) - E_0\| \le \delta$$

for any $(S^0, I^0) \in X_0$, we have

$$\limsup_{m\to\infty} d[P^m(S^0, I^0), E_0] \ge \delta.$$

Proof We know that when $R_0 > 1$, $R_0 - \frac{\bar{\lambda}\eta}{d+\epsilon+\mu+\gamma} - k\eta > 1$ for a small enough η. Now, we proceed by contradiction to prove that

$$\limsup_{m\to\infty} d(P^m(S^0, I^0), E_0) \ge \delta.$$

If not, then

$$\limsup_{m\to\infty} d(P^m(S^0, I^0), E_0) < \delta$$

for some $(S^0, I^0) \in X_0$. Without loss of generality, we assume that $d(P^m(S^0, I^0), E_0) < \delta$ for all $m \ge 0$. By the continuity of the solutions with respect to the initial values, we obtain that

$$\|u(t, P^m(S^0, I^0)) - u(t, E_0)\| < \eta, \forall m \ge 0,\ \forall t_1 \in [0, \omega].$$

For any $t \ge 0$, let $t = m\omega + t_1$, where $t_1 \in [0, \omega]$ and $m = [\frac{t}{\omega}]$, which is the greatest integer less than or equal to $\frac{t}{\omega}$. Then, we have

$$\begin{aligned}&\|u(t, (S^0, I^0)) - u(t, E_0)\|\\ &\quad = \|u(t_1, P^m(S^0 I^0)) - u(t_1, E_0)\| < \eta\end{aligned}$$

for any $t \ge 0$, which implies that $S_0 - \eta < S(t) < S_0 + \eta$ and $0 \le I(t) < \eta$. Then, for $\|(S^0, I^0) - E_0\| < \delta$, we have

$$\frac{dI}{dt} \geq \frac{\lambda(t)(S_0-\eta)I}{1+k\eta} - (d+\varepsilon+\mu)I - \gamma I. \tag{4}$$

Since $R_0 - \frac{\bar{\lambda}\eta}{d+\epsilon+\mu+\gamma} - k\eta > 1$, $I(t) \to \infty$ for $t \to \infty$, which leads to a contradiction. □

The proof of the lemma is completed.

Theorem 2.4 *System (1) has at least one positive periodic solution.*

Proof We first prove that $\{P^m\}_{m\geq 0}$ is uniformly persistent with respect to $(X_0, \partial X_0)$. First of all, we explain that X_0 and ∂X_0 are positively invariant. From the equations in system (2), we can find easily that X_0 is positively invariant. Clearly, ∂X_0 is relatively closed in X. Set

$$M_\partial = \left\{(S^0, I^0) \in \partial X_0 : P^m(S^0, I^0) \in \partial X_0, \forall m \geq 0\right\}.$$

It is easy to show that

$$M_\partial = \{(S, 0) \in X : S \geq 0\}. \tag{5}$$

Note that $\{(S, 0) \in X : S \geq 0\} \subseteq M_\partial$. We only need to prove that

$$M_\partial \subseteq \{(S, 0) \in X : S \geq 0\}.$$

That is, for any $(S^0, I^0) \in \partial X_0$, we have $I(m\omega) = 0, \forall m \geq 0$. If there exists an $m_1 \geq 0$ such that $I(m_1\omega) > 0$, by replacing the initial time 0 with $m_1\omega$, it can be seen that

$$I(t) = I(m_1\omega)e^{\int_{m_1\omega}^{t} \frac{\lambda(t)S(t)}{1+kI(t)} - (d+\varepsilon+\mu) - \frac{\gamma}{1+\alpha I(t)}dt} > 0, \tag{6}$$

which contradicts $(S^0, I^0) \in \partial X_0$ that requires $P^m(S^0, I^0) \in \partial X_0$, for all $m \geq 0$. So, the equality (5) holds, which implies that E_0 is the only fixed point of P and acyclic in ∂X_0. □

Moreover, Lemma 2.3 implies that $E_0 = (S_0, 0)$ is an isolated invariant set in X and $W^S(E_0 \cap X_0) = \emptyset$. By the acyclicity theorem on uniform persistence for maps (Theorem 1.3.1 and Remark 1.3.1 in [47]), it follows that P is uniformly persistent with respect to $(X_0, \partial X_0)$.

Now, Theorem 1.3.6 in [47] implies that P has a fixed point

$$(S^*(0), I^*(0)) \in X_0.$$

From the first equation of system (2), we have that

$$\begin{aligned} S^*(t) &= e^{-\int_0^t (d+\lambda(t)I^*(t))dt} \\ &\quad \times \left[S^*(0) + \int_0^t A e^{\int_0^t (d+\lambda(t)I^*(t))dt} dt\right] \\ &> Ae^{-\int_0^t (d+\lambda(t)I^*(t))dt \int_0^t e^{\int_0^t (d+\lambda(t)I^*(t))dt} dt} \\ &> 0, \ \forall t \in [0, \omega]. \end{aligned}$$

The periodicity of $S^*(t)$ implies $S^*(t) > 0$ for all $t > 0$. Similarly,

$$I^*(t) = I^*(0)e^{\int_0^t \frac{\lambda(t)S^*(t)}{1+kI^*} - (d+\varepsilon+\mu) - \frac{\gamma}{1+\alpha I^*}dt} > 0. \tag{7}$$

Therefore, $(S^*(t), I^*(t))$ is a positive ω-periodic solution of system (1).

Theorem 2.5 *Assume that the following conditions hold:*

$$(H1)\ \frac{\lambda^u \frac{A}{d} - (d+\epsilon+\mu)}{k(d+\epsilon+\mu)} < \frac{A}{d}, \tag{8}$$

$$\begin{aligned} (H2)\ &\left[-\frac{\lambda^l \alpha A}{d+\lambda^u/k} + (d+\epsilon+\mu)(k+\alpha) + \gamma k\right]^2 \\ &> 4\left(d+\epsilon+\mu+\gamma - \frac{\lambda^l \alpha A}{d+\lambda^u/k}\right)k\alpha(d+\epsilon+\mu), \end{aligned} \tag{9}$$

where $\lambda^u = \max_{t\in[0,\omega]} \lambda(t)$ *and* $\lambda^l = \min_{t\in[0,\omega]} \lambda(t)$. *System (2) has at least two positive periodic solutions.*

Proof Firstly, through the transformation of variables

$$u_1(t) = \ln S(t), \quad u_2(t) = \ln I(t),$$

we consider the following system

$$\begin{cases} \dot{u_1}(t) = Ae^{-u_1(t)} - d - \frac{\lambda(t)e^{u_2(t)}}{1+ke^{u_2(t)}}, \\ \dot{u_2}(t) = \frac{\lambda(t)e^{u_1(t)}}{1+ke^{u_2(t)}} - (d+\epsilon+\mu) - \frac{\gamma}{1+\alpha e^{u_2(t)}}. \end{cases} \tag{10}$$

It is easy to know that if $(u_1^*(t), (u_2^*(t))^T$ is a ω-periodic solution of system (10), then $(S^*(t), I^*(t)) = (e^{u_1^*(t)}, e^{u_2^*(t)})^T$ is a positive ω-periodic solution of system (2). So, we only need to consider the existence of ω-periodic solution of the system (10). The rest of the proof follows as Theorem 3.1 in [48]. So, the detailed proof is omitted here. □

3 Numerical simulations of periodic solution

In this section, we first study the stability of the periodic solution of (2) with the given parameter in Table 1 and a

small seasonal fluctuation in transmission rate, which are proved by numerical simulations. Taking $\lambda(t) = a + \eta\sin(\pi t/6)$ values into the system (2), we can get the following equations:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = A - \mathrm{d}S - \frac{(a+\eta\sin(\pi t/6))SI}{1+kI}, \\ \frac{dI}{\mathrm{d}t} = \frac{(a+\eta\sin(\pi t/6))SI}{1+kI} - (d+\epsilon+\mu)I - \frac{\gamma I}{1+\alpha I}. \end{cases} \tag{11}$$

Because the value of η is very small, we can regard η as a small parameter, that is, a small seasonal fluctuation. Let us investigate the existence and stability of periodic solution for system (11). We can apply the singular perturbation approach to assume positive periodic solution to be the form

$$\begin{aligned} &S^*(t,\eta) = S_0(t) + \eta S_1(t) + \eta^2 S_2(t) + \cdots, \\ &I^*(t,\eta) = I_0(t) + \eta I_1(t) + \eta^2 I_2(t) + \cdots, \\ &S_n(t+12) = S_n(t),\ I_n(t+12) = I_n(t),\ n = 0,1,2,\ldots. \end{aligned} \tag{12}$$

Now in order to obtain the periodic solution, we substitute (12) into (11). Comparing the coefficients of the terms η^n, $n = 1, 2, \ldots$, we obtain the following equations

$$\begin{cases} \frac{\mathrm{d}S_0}{\mathrm{d}t} = A - \mathrm{d}S_0 - \frac{aS_0I_0}{1+kI_0}, \\ \frac{\mathrm{d}I_0}{\mathrm{d}t} = \frac{aS_0I_0}{1+kI_0} - (d+\epsilon+\mu)I_0 - \frac{\gamma I_0}{1+\alpha I_0}. \end{cases} \tag{13}$$

and

$$\begin{cases} \frac{\mathrm{d}S_n}{\mathrm{d}t} = x_{n1}S_n + x_{n2}I_n + f_n(t), \\ \frac{\mathrm{d}I_n}{\mathrm{d}t} = x_{n3}S_n + x_{n4}I_n + g_n(t), \quad n = 1,2,\ldots, \end{cases} \tag{14}$$

where

$$\begin{aligned} x_{n1} &= -\frac{1+kI_0+aI_0}{1+kI_0}, \quad x_{n2} = \frac{Ak - \mathrm{d}kS_0 - aS_0}{1+kI_0}, \\ x_{n3} &= \frac{aI_0}{1+kI_0}, \\ x_{n4} &= \frac{aS_0 - (d+\epsilon+\mu)kI_0}{1+kI_0} - \frac{(d+\epsilon+\mu)\alpha I_0 + \gamma}{1+\alpha I_0} \\ &\quad - (d+\epsilon+\mu) - \frac{\gamma kI_0 - 2aS_0I_0}{(1+kI_0)(1+\alpha I_0)}, \\ f_n(t) &= -\frac{1}{1+kI_0}\left[(a+\mathrm{d}k)\sum_{i=1}^{n-1} S_{n-i}(t)I_i \right. \\ &\quad \left. + \sin\frac{\pi t}{6}\sum_{i=0}^{n-1} S_{n-i-1}(t)I_i + k\sum_{i=1}^{n-1}\dot{S}_{n-i}(t)I_i\right], \end{aligned}$$

$$\begin{aligned} g_n(t) &= \frac{1}{(1+kI_0)(1+\alpha I_0))} \\ &\quad \times \left\{ a\sum_{j=1}^{n-1}\sum_{i=0}^{n-j} S_jI_iI_{n-i-j} + a(1+\alpha I_0) \right. \\ &\quad \times \sum_{i=1}^{n-1} S_iI_{n-i} + \sin\frac{\pi t}{6}\left[\sum_{j=0}^{n-2}\sum_{i=0}^{n-j} S_jI_iI_{n-i-j} \right. \\ &\quad \left. + (1+\alpha I_0)\sum_{i=0}^{n-1} S_iI_{n-i}\right] \\ &\quad - (d+\epsilon+\mu)\left[k\alpha\sum_{j=0}^{n-1}\sum_{i=1}^{n-j-1} I_jI_iI_{n-i-j} \right. \\ &\quad \left. + ((1+\alpha I_0)k + (1+kI_0)\alpha)\sum_{i=1}^{n-1} I_iI_{n-i}\right] \\ &\quad - \gamma k\sum_{i=1}^{n-1} I_iI_{n-i} - k\alpha\sum_{j=0}^{n-1}\sum_{i=1}^{n-j-1} I_iI_{n-i-j}\dot{I}_j \\ &\quad \left. - [(1+\alpha I_0)k + (1+kI_0)\alpha]\sum_{i=1}^{n-1} I_i\dot{I}_{n-i}\right\}. \end{aligned}$$

With regard to the system (13), let $k = 0.01, \alpha = 1, \epsilon = 0.5, A = 50, d = 0.02, a = 2\times10^{-4}, \eta = 3\times10^{-5}, \mu = 0.07$, and $\gamma = 0.5$; we can give its periodic solution:

$$\begin{aligned} &(S_0(t), I_0(t)) = (S^*, I^*) \\ &= (4894.0608591, 12.24589776651). \end{aligned} \tag{15}$$

Substituting (15) into (14) for $n = 1$, it is can be easily obtained that

$$\begin{cases} \frac{\mathrm{d}S_1}{\mathrm{d}t} = x_{11}S_1 + x_{12}I_1 - \frac{\sin\frac{\pi t}{6}S_0I_0}{1+kI_0}, \\ \frac{\mathrm{d}I_1}{\mathrm{d}t} = x_{13}S_1 + x_{14}I_1 + \frac{\sin\frac{\pi t}{6}(1+\alpha I_0)S_0I_0}{1+kI_0}, \quad n = 1,2,\ldots. \end{cases} \tag{16}$$

It can be easily known that (16) has a unique periodic solution for given initial values. Moreover, since $f_n(t)$ and $g_n(t)$ are continuous and bounded functions, we can obtain that the solution of (14) exists uniquely on $[0,\infty]$ for given initial values by the mathematical induction when $n = 2, 3, \ldots$.

Next, we investigate the stability of the periodic solution $S_p(t,\eta), I_p(t,\eta)$ by analyzing the perturbation for the periodic solution. Applying the transformation $S = S_p(t,\eta) + s, I = I_p(t,\eta) + i$, we can obtain

$$\begin{cases} \frac{ds}{dt} = -ds + \frac{(a+\eta\sin(\pi t/6))S_pI_p}{1+kI_p} \\ \quad -\frac{(a+\eta\sin(\pi t/6))(S_p+s)(I_p+i)}{1+k(I_p+i)}, \\ \frac{di}{dt} = -\frac{(a+\eta\sin(\pi t/6))S_pI_p}{1+kI_p} \\ \quad +\frac{(a+\eta\sin(\pi t/6))(S_p+s)(I_p+i)}{1+k(I_p+i)} \\ \quad -(d+\epsilon+\mu)i + \frac{\gamma I_p}{1+\alpha I_p} - \frac{\gamma(I_p+i)}{1+\alpha(I_p+i)}. \end{cases} \tag{17}$$

It can be obtained by expanding the function $\frac{(a+\eta\sin(\pi t/6))(S_p+s)(I_p+i)}{1+k(I_p+i)}$, $\frac{\gamma(I_p+i)}{1+\alpha(I_p+i)}$ about $(s,i) = (0,0)$ by Taylor formula in the above system:

$$\begin{cases} \frac{ds}{dt} = -\left(d + \frac{(a+\eta\sin(\pi t/6))I_p}{1+kI_p}\right)s \\ \quad -\frac{(a+\eta\sin(\pi t/6))S_p}{(1+kI_p)^2}i - o(\rho), \\ \frac{di}{dt} = \frac{(a+\eta\sin(\pi t/6))I_p}{1+kI_p}s + \left(\frac{(a+\eta\sin(\pi t/6))S_p}{(1+kI_p)^2}\right. \\ \quad \left. -(d+\epsilon+\mu) - \frac{\gamma}{(1+\alpha I_p)^2}\right)i + o(\rho), \end{cases} \tag{18}$$

where $\rho = \sqrt{s^2+i^2}$. Moreover, we expand $\frac{(a+\eta\sin(\pi t/6))I_p}{1+kI_p}$, $\frac{(a+\eta\sin(\pi t/6))S_p}{(1+kI_p)^2}$, and $\frac{\gamma}{(1+\alpha I_p)^2}$ about $\eta = 0$ by Taylor formula and substitute them into system (18) to obtain

$$\begin{cases} \frac{ds}{dt} = -\left(d + \frac{aI_0}{1+kI_0}\right)s - \frac{aS_0}{(1+kI_0)^2}i \\ \quad -\eta(b_1s + b_2i) - o(\eta)(s+i) - o(\rho), \\ \frac{di}{dt} = \frac{a+I_0}{1+kI_0}s + \left(\frac{aS_0}{(1+kI_0)^2}\right. \\ \quad \left. -(d+\epsilon+\mu) - \frac{\gamma}{(1+\alpha I_0)^2}\right)i \\ \quad +\eta(b_1s + (b_2+b_3)i) + o(\eta)(s+i) + o(\rho), \end{cases} \tag{19}$$

where $b_1 = \frac{I_0\sin(\pi t/6)+aI_1+aI_0^2\sin(\pi t/6)}{(1+kI_0)^2}$, $b_1 = \frac{(S_0\sin(\pi t/6)+aS_1)(1+kI_0)-2aS_0(1+kI_0)kI_1}{(1+kI_0)^4}$, and $b_3 = \frac{2\alpha\gamma I_1(1+\alpha I_0)}{(1+\alpha I_0)^4}$.

Constructing the Lyapunov function as follows,

$$V(t) = |s(t)| + |i(t)|, \tag{20}$$

we can calculate the upper right derivative of $V(t)$ along the solution of (19)

$$\begin{aligned} D^+V(t) &\le \mathrm{sign(s)}\dot{s} + \mathrm{sign(i)}\dot{i} \\ &\le -d|s| - ((d+\epsilon+\mu) \\ &\quad + \frac{\gamma}{(1+\alpha I_0)^2} - \frac{2aS_0}{(1+kI_0)^2} - 2b_2\eta \\ &\quad - b_3\eta - 2o(\eta))|i| + 2|o(\rho)|, \end{aligned}$$

where $(d+\epsilon+\mu) + \frac{\gamma}{(1+\alpha I_0)^2} - \frac{2aS_0}{(1+kI_0)^2} = 0.1399 > 0$. Thus, for a sufficiently small η, there exists $c > 0$ such that

$$D^+V(t) < cV(t) + 2|o(\rho)|, \tag{21}$$

where $c < \min\{d, (d+\epsilon+\mu) + \frac{\gamma}{(1+\alpha I_0)^2} - \frac{2aS_0}{(1+kI_0)^2} - 2b_2\eta - b_3\eta - 2o(\eta)\}$ which indicates in any sufficient neighborhood of the origin on $s-i$ plane,

$$\begin{aligned} &V(t) \le V(0), \forall t \ge 0, \\ &\lim_{t\to\infty}|s(t)| = 0, \lim_{t\to\infty}|i(t)| = 0. \end{aligned} \tag{22}$$

So, $S_p(t,\eta)$, $I_p(t,\eta)$ is stable for a sufficiently small η.

Now, we further investigate the dynamical behaviors of (2) by numerical simulations:

(1) When $R_0 < 1$ and $R_0^c < 1$, the disease will disappear which can be seen in Fig. 2.
(2) When $R_0 < 1$ and $k > \alpha$, the disease will disappear which can be seen in Fig. 3.
(3) When $R_0 > 1$, the disease-free equilibrium become unstable and there is at least a positive periodic solution which can be seen in Fig. 4.
(4) $R_0 < 1$ cannot promise the global stability of disease-free equilibrium and it needs to add other conditions which can be seen in Fig. 5.

In this case, there can exist two positive solutions where one is stable and the other is unstable which can be seen in Fig. 6a, b.

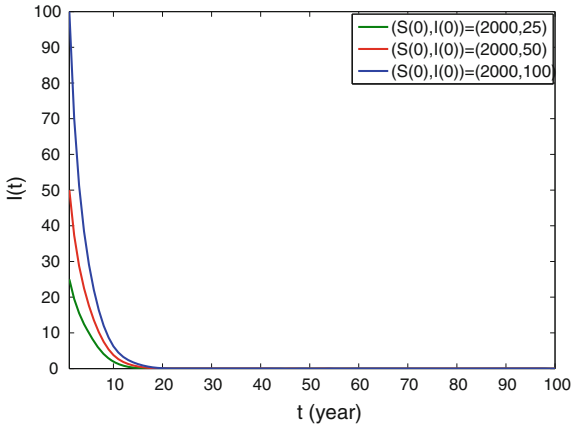


Fig. 2 The solution curves of system (1) with time when $k = 0.01$, $\alpha = 1$, $\epsilon = 0.5$, $A = 50$, $d = 0.02$, $a = 2\times10^{-4}$, $\eta = 3\times10^{-5}$, $\mu = 0.07$, and $\gamma = 0.5$. Here, $R_0 = 0.5025$ and $R_0^c = 0.9284$

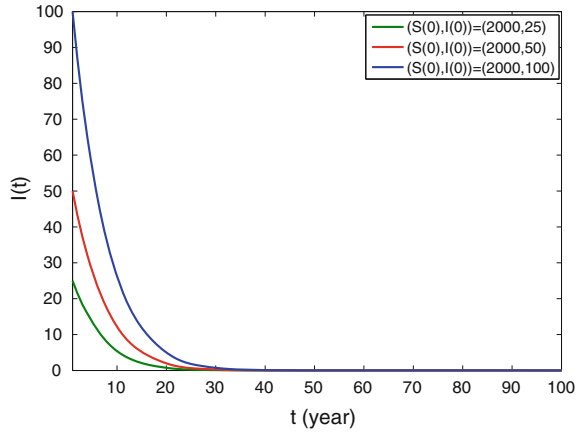


Fig. 3 The solution curves of system (1) with time when $k = 0.5, \alpha = 0.4, \epsilon = 0.05, A = 50, d = 0.02, a = 2 \times 10^{-4}, \eta = 3 \times 10^{-5}, \mu = 0.07$, and $\gamma = 0.5$. Here, $R_0 = 0.8559$ and $R_0^c = 3.9125$

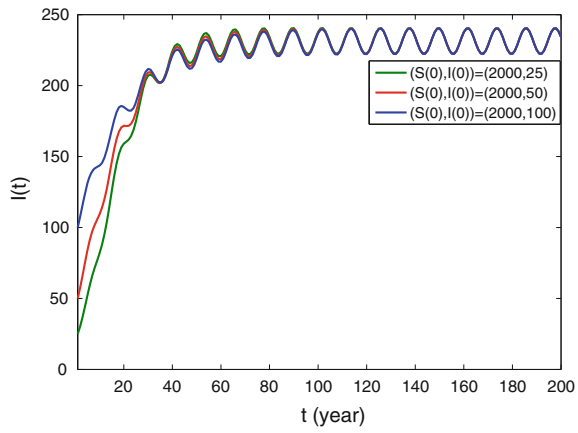


Fig. 4 The solution curves of system (1) with time when $k = 0.01, \alpha = 1, \epsilon = 0.05, A = 80, d = 0.02, a = 2 \times 10^{-4}, \eta = 3 \times 10^{-5}, \mu = 0.07$, and $\gamma = 0.5$. Here, $R_0 = 1.3694$

Let $\eta = 0.0003$ and keep other parameters unchanged as above; it can be obtained by numerical calculations that

$$\frac{\lambda^u \frac{A}{d} - (d+\epsilon+\mu)}{k(d+\epsilon+\mu)} = 155.3571, \quad \frac{A}{d} = 2500,$$

$$\left[-\frac{\lambda^l \alpha A}{d+\lambda^u/k} + (d+\epsilon+\mu)(k+\alpha) + \gamma k\right]^2 = 0.0026,$$

$$2\sqrt{\left(d+\epsilon+\mu+\gamma-\frac{\lambda^l \alpha A}{d+\lambda^u/k}\right) k\alpha(d+\epsilon+\mu)} = 0.0025,$$

which satisfy all the conditions in Theorem 2.5. Thus, system (2) has at least two positive 12-periodic solutions. We also know that the domains of $S(t)$ and $I(t)$ are

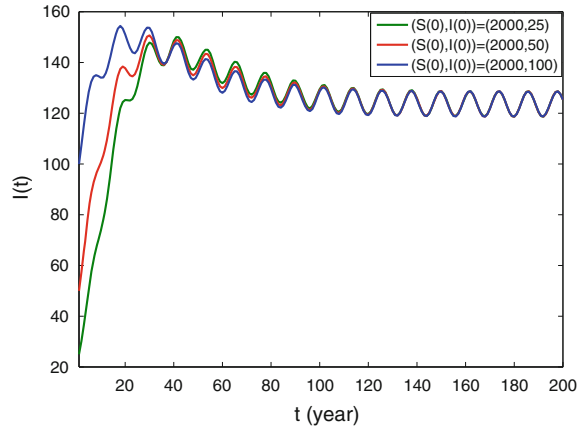


Fig. 5 The solution curves of system (1) with time when $k = 0.01, \alpha = 1, \epsilon = 0.05, A = 50, d = 0.02, a = 2 \times 10^{-4}, \eta = 3 \times 10^{-5}, \mu = 0.07$, and $\gamma = 0.5$. Here, $R_0 = 0.8559$

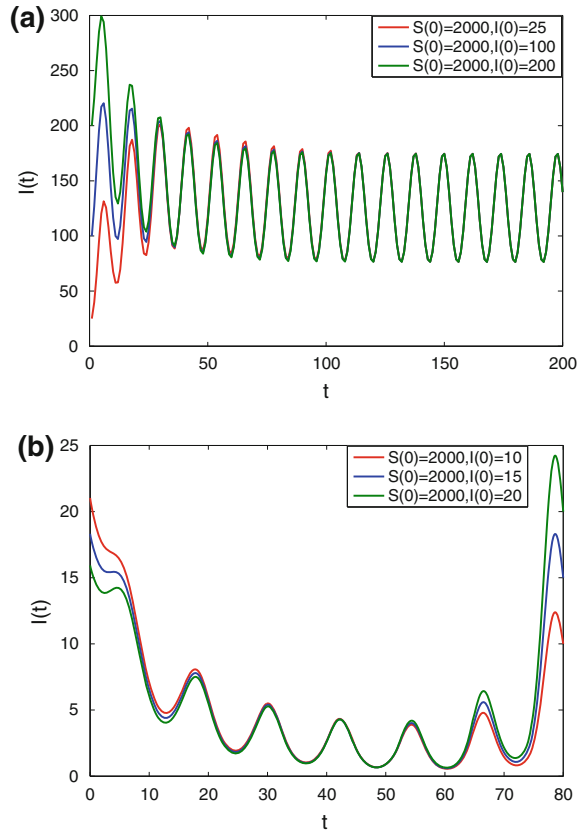


Fig. 6 The solution curves of system (2) under different initial conditions. **a** The positive curve $I(t)$ of solutions starting in M_2. **b** The negative curve of solutions in M_1

$$M_1 := \{(S(t), I(t)) | 1162.8 < S(t) < 2500, 0.1494 < I(t) < 13.9085\},$$

$$M_2 := \{(S(t), I(t)) | 1162.8 < S(t) < 2500, 22.7161 < I(t) < 310.7143\}.$$

According to the above discussion, the periodic solution in M_2 is stable, which can be seen in Fig. 2. The top figure demonstrates that all solutions starting in M_2 tend toward a periodic solution, that is, the periodic solution in M_2 is stable. The other positive periodic solution of system (2) is not easy to obtain numerically. The forward solutions starting from M_1 will leave M_1 and tend toward the periodic solution in M_2 or zero. So, we reverse the time to simulation, that is, to obtain the backward solution of system (2) with initial values in M_1, which is the bottom figure in Fig. 2. This demonstrates that the solutions starting in the neighborhood of the periodic solution in M_1 will tend toward this periodic solution firstly and then deviate from it, which indicates that the periodic solution in M_1 is unstable.

4 Discussion

In this paper, we consider a nonautonomous system with periodic transmission rate, whose corresponding autonomous system has been discussion in [37]. It is concluded that the basic reproduction number being the unity is a strict threshold for disease eradication when the effect of delayed treatment is weak, that is, $\alpha < (d + \epsilon + \mu + \gamma)(\mathrm{d}k + \lambda)/(\mathrm{d}\gamma)$. However, when this effect is strong, there exists a backward bifurcation. In this case, letting the basic reproduction number be below one is not enough to eradicate the disease. Based on this consideration of saturated treatment, we add the periodic incidence rate to the model. Firstly, we find that the basic reproduction number less than 1 cannot keep the global stability of disease-free equilibrium and it needs tighter conditions such as $R_0^c < 1$. Moreover, by Theorem 2.5 and Fig. 2, we can observe that there are two periodic solutions for system (2) where one is stable and the other is unstable. These theories will provide some guidance for control measures of diseases.

In practice, we need to take effective control measures to keep $R_0^c < 1$ not $R_0 < 1$. Moreover, there can exist a periodic solution in our model even if $R_0 < 1$. So, we can adopt some prevention measures before the peak is coming according to specific disease cycles and factors for cycles. In addition, as shown in Fig. 6, when the initial conditions are different, the disease will tend toward different periodic solutions. So, besides related control measures, we can change the initial condition to change the tendency of the disease. Our future research will focus on the stability of the periodic solution and apply our mathematic methods to the research of special diseases.

Acknowledgments The research was partially supported by the National Natural Science Foundation of China under Grant Nos. 11301490, 11331009, 11147015, and 11171314, the Natural Science Foundation of Shan'Xi Province Grant Nos. 2012021002-1 and 2012011002-2.

References

1. CDC: http://www.cdc.gov/pertussis/about/index.html (2011). Accessed 5 Feb 2011
2. WHO: http://www.who.int/topics/measles/en/ (2011). Accessed 5 Feb 2011
3. WHO: http://www.who.int/ith/diseases/measles/en/ (2011). Accessed 5 Feb 2011
4. WHO: http://www.who.int/influenza/human_animal_interface/en/ (2011). Accessed 5 Feb 2011
5. WHO: http://www.euro.who.int/en/what-we-do/health-topics/communicable-diseases/influenza/news2/news/2010/12/start-of-influenza-season (2011). Accessed 5 Feb 2011
6. CDC: http://www.cdc.gov/rabies/ Animal bites present a risk for rabies (2011). Accessed 5 Feb 2011
7. Koelle, K., Pascual, M., Yunus, M.: Pathogen adaptation to seasonal forcing and climate change. Proc. R. Soc. B **272**, 971–977 (2005)
8. Shek, L.P.-C., Lee, B.-W.: Epidemiology and seasonality of respiratory tract virus infections in the tropics. Paediatr. Respir. Rev. **4**, 105–111 (2003)
9. Liu, J.L.: Threshold dynamics for a HFMD epidemic model with periodic transmission rate. Nonlinear Dyn. **64**, 89–95 (2011)
10. Bowong, S.: Optimal control of the transmission dynamics of tuberculosis. Nonlinear Dyn. **61**, 729–748 (2010)
11. Khallaayoune, K., Laamrani, H.: Seasonal patterns in the transmission of Schistosoma haematobium in Attaouia, Morocco. J. Helminthol. **66**, 89–95 (1992)
12. Song, M., Tang, Q., Wang, D.-M., Mo, Z.-J., Guo, S.-H., Li, H., Tao, X.-Y., Rupprecht, C.E., Feng, Z.-J., Liang, G.-D.: Epidemiological investigations of human rabies in China. BMC Infect. Dis. **9**, 210 (2009)
13. Jong, J.G.D., Winkler, K.C.: Survival of measles virus in air. Nature **201**, 1054–1055 (1964)
14. Grassly, N.C., Fraser, C.: Seasonal infectious disease epidemiology. Proc. R. Soc. B **273**, 2541–2550 (2006)
15. Hethcote, H.W., Stech, H.W., Driessche, P.v.d.: Periodicity and stability in epidemic models: a survey. In: Busenberg, S., Cooke, K.L. (eds.) Differential Equations and Applications in Ecology, Epidemics and Populations Problems, pp. 65–82. Academic Press, New York (1981).
16. Liu, W.M., Hethcote, H.W., Levin, S.A.: Dynamical behavior of epidemiological models with nonlinear incidence rates. J. Math. Biol. **25**, 359–380 (1987)

17. Yuan, Z., Ma, Z., Tang, X.: Global stability of a delayed HIV infection model with nonlinear incidence rate. Nonlinear Dyn. **68**, 207–214 (2012)
18. Bowong, S., Kurths, J.: Modeling and analysis of the transmission dynamics of tuberculosis without and with seasonality. Nonlinear Dyn. **67**, 2027–2051 (2012)
19. London, W., Yorke, J.A.: Recurrent outbreaks of measles, chickenpox and mumps. I. Seasonal variation in contact rates. Am. J. Epidemiol. **98**, 453–468 (1973)
20. Dowell, S.F.: Seasonal variation in host susceptibility and cycles of certain infectious diseases. Emerg. Infect. Dis. **7**, 369–374 (2001)
21. Bjornstad, O.N., Finkenstadt, B.F., Grenfell, B.T.: Dynamics of measles epidemics: estimating scaling of transmission rates using a time series SIR model. Ecol. Monogr. **72**, 169–184 (2002)
22. Billings, L., Schwartz, I.B.: Exciting chaos with noise: unexpected dynamics in epidemic outbreak. J. Math. Biol. **44**, 31–48 (2002)
23. Schwartz, I.B.: Small amplitude, long periodic out breaks in seasonally driven epidemics. J. Math. Biol. **30**, 473–491 (1992)
24. Schwartz, I.B., Smith, H.L.: Infinite subharmonic bifurcation in an SIER epidemic model. J. Math. Biol. **18**, 233–253 (1983)
25. Smith, H.L.: Multiple stable subharmonics for a periodic epidemic model. J. Math. Biol. **17**, 179–190 (1983)
26. Dushoff, J., Plotkin, J.B., Levin, S.A., Earn, D.J.D.: Dynamical resonance can account for seasonality of influenza epidemics. Proc. Natl. Acad. Sci. **101**, 16915–16916 (2004)
27. Gao, S., Liu, Y., Nieto, J.J., Andrade, H.: Seasonality and mixed vaccination strategy in an epidemic model with vertical transmission. Math. Comput. Simul. **81**, 1855–1868 (2011)
28. Zhou, L., Fan, M.: Dynamics of an SIR epidemic model with limited medical resources revisited. Nonlinear Anal Real World Appl. **13**, 312–324 (2012)
29. Cushing, J.M.: A juvenile–adult model with periodic vital rates. J. Math. Biol. **53**, 520–539 (2006)
30. Earn, D.J.D., Rohani, P., Bolker, B.M., Grenfell, B.T.: A simple model for complex dynamical transitions in epidemics. Science **287**, 667–670 (2000)
31. Keeling, M.J., Rohani, P., Grenfell, B.T.: Seasonally forced disease dynamics explored as switching between attractors. Physica D **148**, 317–335 (2001)
32. Aron, J.L., Schwartz, I.B.: Seasonality and period-doubling bifurcations in an epidemic model. J. Theor. Biol. **110**, 665–679 (1984)
33. Capasso, V., Serio, G.: A generalization of the Kermack-McKendrick deterministic epidemic model. Math. Biosci. **42**, 43–61 (1978)
34. Zhou, Y.G., Xiao, D.M., Li, Y.L.: Bifurcations of an epidemic model with non-monotonic incidence rate of saturated mass action. Chaos Solitons Fractals **32**, 1903–1915 (2007)
35. Xiao, D.M., Ruan, S.G.: Global analysis of an epidemic model with nonmonotone incidence rate. Math. Biosci. **208**, 419–429 (2007)
36. Zhang, T.L., Teng, Z.D.: Pulse vaccination delayed SEIRS epidemic model with saturation incidence. Appl. Math. Model. **32**, 1403–1416 (2008)
37. Zhang, X., Liu, X.N.: Backward bifurcation of an epidemic model with saturated treatment function. J. Math. Anal. Appl. **348**, 433–443 (2008)
38. Zhang, Z.H., Suo, Y.H.: Qualitative analysis of a SIR epidemic model with saturated treatment rate. J. Appl. Math. Comput. **34**, 177–194 (2010)
39. Bacaer, N., Guernaoui, S.: The epidemic threshold of vector-borne diseases with seasonality. J. Math. Biol. **53**, 421–436 (2006)
40. Bacaer, N.: Approximation of the basic reproduction number R_0 for vector-borne diseases with a periodic vector population. Bull. Math. Biol. **69**, 1067–1091 (2007)
41. Bacaer, N., Abdurahman, X.: Resonance of the epidemic threshold in a periodic environment. J. Math. Biol. **57**, 649–673 (2008)
42. Bacaer, N., Dads, E.H.A.: Genealogy with seasonality, the basic reproduction number, and the influenza pandemic. J. Math. Biol. **62**, 741–762 (2011)
43. Bacaer, N.: Periodic matrix population models: growth rate, basic reproduction number, and entropy. Bull. Math. Biol. **71**, 1781–1792 (2009)
44. Wang, W., Zhao, X.-Q.: Threshold dynamics for compartmental epidemic models in periodic environments. J. Dyn. Differ. Equ. **20**, 699–717 (2008)
45. Lakshmikantham, V., Leela, S., Martynyuk, A.A.: Stability Analysis of Nonlinear Systems. Marcel Dekker Inc., New York (1989)
46. Perko, L.: Differential Equations and Dynamical Systems. Springer, New York (2000)
47. Zhao, X.-Q.: Dynamical Systems in Population Biology. Springer, New York (2003)
48. Bai, Z.G., Zhou, Y.C., Zhang, T.L.: Existence of multiple periodic solutions for an SIR model with seasonality. Nonlinear Anal. **74**, 3548–3555 (2011)