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A Normalized Numerical Scaling Method for the Unbalanced Multi-Granular Linguistic Sets

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Decision makers often express their evaluations on decision problems with multi-granular linguistic terms. This fact leads to the unification of the multi-granular linguistic terms into a single linguistic set in the literature. However, this unification process increases the complexity of computation and the subjectivity in the determination of transformation functions. To overcome this deficiency, this paper aims to develop a normalized numerical scaling method for determining the semantics of multi-granular linguistic terms in the same domain. We first introduce a class of numerical scaling functions to generate several balanced or unbalanced linguistic sets. Since these scaled linguistic sets have different domains, we then develop a normalized numerical scaling method to form them into the unique interval [0,1]. As a result of this development, two classes of normalized scaling functions are derived from the priori scale information and applications of piecewise linear interpolation and piecewise arc interpolation. Finally, an example is given to illustrate how the method works.

Keywords: Unbalanced; multi-granular linguistic sets; normalized scaling function; piecewise interpolation; 2-tuple linguistic.

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1. Introduction

In many real decision making (DM) problems, it is more convenient to assess the aspects of alternatives by means of the natural language than by precise numerical values.[1–7] When a DM problem is solved by linguistic information, it implies the need for a linguistic computational model.[8–11] Four linguistic computational models are deployed in the decision making field:[12]

(1) linguistic computational model based on membership functions;[13–16]
(2) linguistic computational model based on type-2 fuzzy sets;[17–20]
(3) linguistic computational model based on ordinal scales;[21–26]
(4) 2-tuple linguistic computational model.[2,11,27–32]

The linguistic computational models can also be roughly categorized into membership functions based models[13–20] and symbolic linguistic computational models.[11,21–24,26,28,29] The first two listed above are membership functions based models and the latter two models are symbolic computational models.[33] Due to the simplicity and understandability, symbolic linguistic computational models have received more researchers' attentions.[2,25,27,30,31] This study devotes to the symbolic linguistic models, especially the 2-tuple linguistic computational model that is proposed by Herrera and Matínez in Ref. 2.

Decision makers in group or multi-stage decision making problems usually provide their preferences over alternatives in the linguistic domains with different granularities for their different knowledge backgrounds, judging abilities or cognitive levels.[25] The multi-granular linguistic provides a flexible way for decision makers to express their preferences.[15,28,34–37] The model of multi-granular linguistic was originated by Herrera *et al.*[34] Thereafter, many researchers have successfully enhanced and applied the model to solve the practical decision making problems. Herrera and Martínez[28] defined a set of multi-granular linguistic contexts and developed some transformation functions based on 2-tuple linguistic representation model. Cordón *et al.*[35] proposed a hierarchical system of linguistic rules-learning methodology. Herrera-Viedma *et al.*[36] designed a multi-granular linguistic based information retrieval system to facilitate the interchange of information between the user and the system. Jiang *et al.*[15] proposed a linear goal programming method to solve the group decision making problems with multi-granular linguistic information under the membership function based linguistic computation model. Mata *et al.*[37] analyzed the consensus process in group decision making problems under the multi-granular linguistic information and proposed an adaptive consensus support model to reduce the number of consensus rounds.

Although the preference information in many decision making problems is characterized by linguistic terms in a uniformly and symmetrically distributed linguistic set (balanced),[5,6,15,28,34–38] there still are a lot of problems whose assessments are better modeled by means of a non-uniformly or asymmetrically distributed linguistic set (unbalanced).[4,11,25,31] The unbalanced linguistic information appears due to

Fig. 1. The first type of unbalanced linguistic set.

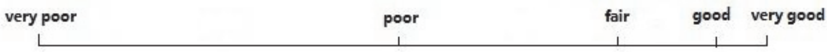

Fig. 2. The second type of unbalanced linguistic set.

either the problems of which it is necessary to assess the preferences with a greater granularity on one side than on the other side, or the nature of the linguistic variables for the evaluation of the problems. According to the literature, there are mainly two types of unbalanced linguistic sets. One type of unbalanced linguistic sets has the characteristic that the linguistic terms are asymmetrically distributed and hence the spacings between adjacent terms may be unequal (namely, the first type of unbalanced linguistic sets, see Fig. 1).[4,11,29,39,40] The other type of unbalanced linguistic sets has the same number of linguistic terms on both sides of the central term but the spacings between adjacent terms may be unequal (namely, the second type of unbalanced linguistic set, see Fig. 2).[1,7,25,27,31,42–44] Herrera *et al.*[11] developed a methodology to deal with the unbalanced linguistic information based on both the concept of linguistic hierarchy and the 2-tuple fuzzy linguistic representation model. Herrera-Viedma et al.[4,29] presented a superior performance information retrieval system that accepts the unbalanced linguistic sets to express weights, and applies linguistic OWA operators to aggregate the unbalanced linguistic information in the consensus models. Cabrerizo *et al.*[39,40] established a consensus model for the complete and incomplete group decision making problems in which decision makers provide their opinions by means of unbalanced linguistic terms. These five papers all concern with the representations and applications of the first type of unbalanced linguistic sets. There are several articles on the second type of unbalanced linguistic sets in the literature. Wang and Hao[31] introduced a propositional 2-tuple linguistic representation model to deal with unbalanced linguistic terms. Dong *et al.*[27,42] constructed optimization models to compute numerical scales of the general unbalanced linguistic sets and AHP unbalanced linguistic sets by defining the concepts of transitive calibration matrix and the consistent index. Xu[25] gave a class of unbalanced linguistic sets with special characteristics, and developed a kind of transformation functions to unify the given multi-granular linguistic terms into a linguistic set. Yu *et al.*[7] proposed another transformation functions, and established several reference tables for making the transformation more convenient. Recently, Cabrerizo et al.[1] used Particle Swarm Optimization (PSO) and granular computing methods to compute the distribution and semantics of the unbalanced linguistic terms. Pedrycz and Song[43,44] used PSO as an optimization environment to determine the most appropriate semantics for multi-granular linguistic terms, they also showed that the unbalanced linguistic sets are usually outperform the balanced ones when considering the inconsistency index of the AHP judgement matrixes. In this study, we will focus on the second type of unbalanced linguistic sets.

To date, there are many studies involving unbalanced linguistic sets, but a formal definition of unbalanced linguistic set is still missing in the literature. The main difference between a balanced linguistic set and an unbalanced linguistic term set is whether the scales of adjacent terms are equidistance or not. This observation leads to our formal definitions of balanced and unbalanced linguistic sets. To tackle the multi-granular linguistic contexts, many researchers utilized transformation functions to unify the multi-granular terms into a linguistic set.[7,25,28] However, the unification process increases both the computing complexity and the subjectivity in the determination of transformation functions. In this study, we propose a normalized numerical scaling method for determining the semantics of multi-granular linguistic terms in the same domain.

In some practical multi-stage decision making problems, it is more convenient for a decision maker to articulate the numerical scales (semantics) of a linguistic set with a smaller granularity. With a better understanding of the problems and the related alternatives, the decision maker would like to use some terms in the linguistic sets with larger granularities in the latter states. Thus, it is an important task to estimate the numerical scales of the linguistic sets with other granularities based on the priori scaled linguistic set. In this paper, the piecewise interpolation method is utilized to construct two kinds of normalized numerical scaling functions for estimating the semantics of the linguistic sets. In order to explain the rationality and validity of the proposed method, we also extend the 2-tuple linguistic model under the circumstance of the scaled multi-granular unbalanced linguistic sets.

The paper is organized as follows. Section 2 reviews some concepts about the symbolic linguistic representations and the classical 2-tuple linguistic representation model. In Sec. 3, we give the concepts of balanced and unbalanced linguistic sets, extend the classical 2-tuple linguistic model to the unbalanced linguistic sets based 2-tuple model, and propose a normalized numerical scaling method to avoid the transformation processes. Section 4 employs the piecewise interpolation to construct two kinds of normalized scaling functions for scaling the linguistic sets with different granularities. Section 5 compares the normalized multi-granular unbalanced linguistic method with other representative unbalanced linguistic representation methods from several aspects. An illustrative example is presented in Sec. 6. Finally, some concluding remarks and further research topics are included in Sec. 7.

2. Preliminaries

Some concepts about the symbolic multi-granular linguistic representation are reviewed in this section.

2.1. *Multi-granular linguistic term sets*

In linguistic decision making, it is a vital task to determine the appropriate linguistic descriptors and semantics for linguistic sets.

In general, a classical discrete linguistic set takes the form of $S_g = \{s_i^{(g)} | i = 0, 1, 2, \ldots, (g-1)\}$, where $s_i^{(g)}$ is a linguistic term and represents a possible value for a linguistic variable.[2,27] Specially, $s_0^{(g)}$ is the lower limit, and $s_{g-1}^{(g)}$ is the upper limit. g $(g > 1)$ is an odd number and represents the granularity of a linguistic set. S_g must have the following characteristics.[11]

(1) The set is ordered: $s_i^{(g)} \geq s_j^{(g)}$ if $i \geq j$.

(2) $s_{\frac{g-1}{2}}^{(g)}$ is the central term.

(3) There is a negation operator: $neg(s_i^{(g)}) = s_j^{(g)}$ such that $j = (g-1) - i$.

(4) Max operator: $max(s_i^{(g)}, s_j^{(g)}) = s_i^{(g)}$, if $s_i^{(g)} \geq s_j^{(g)}$.

(5) Min operator: $min(s_i^{(g)}, s_j^{(g)}) = s_i^{(g)}$, if $s_i^{(g)} \leq s_j^{(g)}$.

For example, a linguistic set of seven terms could be

$$S_7 = \{s_0^{(7)} : \textit{extremely poor}, s_1^{(7)} : \textit{very poor}, s_2^{(7)} : \textit{poor}, s_3^{(7)} : \textit{fair}, s_4^{(7)} : \textit{good}, s_5^{(7)} : \textit{very good}, s_6^{(7)} : \textit{extremely good}\}.$$

Xu[25] put forward a kind of practical linguistic term set as $S'_{2t-1} = \{s_i^{(2t-1)} | i = -\frac{2(t-1)}{t+2-t}, -\frac{2(t-1-1)}{t+2-(t-1)}, \ldots, -\frac{2\times(2-1)}{t+2-2}, 0, \frac{2\times(2-1)}{t+2-2}, \ldots, \frac{2(t-1-1)}{t+2-(t-1)}, \frac{2(t-1)}{t+2-t}\}$ in which t $(t > 1)$ is a positive integer and $2t-1$ is the granularity of the set. S'_{2t-1} is ordered and also has the similar max operator and min operator. Especially, $s_0^{(2t-1)}$ is the central term which represents an assessment of "indifference"; the negation operator is defined as $neg(s_i^{(2t-1)}) = s_{-i}^{(2t-1)}$. The famous AHP linguistic term set takes the form $S^{AHP} = S_{17} = \{s_{\frac{1}{i}} \mid i = 2, 3, \ldots, 9\} \cup \{s_i \mid i = 1, 2, \ldots, 9\}$. And there are also some other linguistic sets with different descriptors and semantics in the literature.[7,24,41,42]

Note that if only one granularity is concerned, the superscript of the terms could be omitted; in case of two or more granularities, the superscript should not be omitted. It is worth pointing out that linguistic sets with smaller granularities are beneficial for expressing the assessments clearly, and the sets with larger granularities are good for delivering exact assessments with more choices.

2.2. *2-tuple linguistic representation model*

Herrera and Martínez[2] proposed a 2-tuple fuzzy linguistic representation model to avoid the loss of information and improve the precision in processes of computing with words. The model represents the linguistic information by means of a pair of values $(s_i, \alpha) \in \overline{S}_g$ $(\overline{S}_g = (S_g - \{s_0, s_{g-1}\}) \times [-0.5, 0.5) \cup \{s_0\} \times [0, 0.5) \cup \{s_{g-1}\} \times [-0.5, 0])$, where $s_i \in S_g$ is a linguistic term and $\alpha \in [-0.5, 0.5)$ is a numerical value that represents the value of the symbolic translation. A function is defined with the purpose of making transformations between linguistic 2-tuples and numerical values.

Definition 1.[2] Let $S_g = \{s_0, s_1, \ldots, s_{g-1}\}$ be a linguistic set and $\beta \in [0, g-1]$ be a value representing the result of a symbolic aggregation operation. A tuple that expresses the equivalent information to β is then obtained as follows:

$$\Delta : \ [0, g-1] \ \to \overline{S}_g \,,$$

$$\Delta(\beta) = (s_i, \alpha) \ with \begin{cases} s_i, & i = round(\beta), \\ \alpha = \beta - i, & \alpha \in [0.5, 0.5)\,, \end{cases} \tag{1}$$

where i is the closet index label to β and α is the value of the symbolic translation.

Clearly, Δ is bijective, and $\Delta^{-1} : \overline{S}_g \to [0, g-1]$ is defined by $\Delta^{-1}(s_i, \alpha) = i + \alpha$. Then, the 2-tuple of $\overline{S}_g$ is identified by the numerical value in $[0, g]$ and the retranslation step is carried out accurately. The computational model for the 2-tuple linguistic could be found in Refs. 2, 4 and 30. Wang and Hao[31] extended the 2-tuple linguistic representation model to the proportional 2-tuple linguistic computational model. Dong *et al.*[27] proposed a new 2-tuple linguistic model by means of the concept of numerical scales. In the following sections, we will improve the 2-tuple linguistic model and use it to explain the decision making results in several illustrative examples.

3. Unbalanced Linguistic Sets and a Normalized Numerical Scaling Method

We introduce the concepts of balanced and unbalanced linguistic sets (the second type) based on numerical scaling functions, and extend the 2-tuple linguistic representation model in Subsec. 3.1. Moreover, we present a normalized numerical scaling method for determining the semantics of multi-granular linguistic terms in the same domain.

3.1. *Balanced and unbalanced linguistic sets*

Definition 2.[27] Let $f : [0, g-1] \to R$ be a monotonic increasing function with R being the real number set. Then, $S_g^f = \{s_{f(0)}, s_{f(1)}, \ldots, s_{f(i)}, \ldots, s_{f(g-1)}\}$ is called a f-scaled linguistic term set, in which $f(i)$ is the numerical scale of the i^{th} term and g $(g > 1)$ is the granularity of the term set.

In most cases, we suppose that a scaling function is strictly monotonic increasing. The frequently used linguistic set $S_g = \{s_0, s_1, \ldots, s_{g-1}\}$ can be seen as a term set scaled by the identity function. S_g^f is a discrete linguistic set with the following characteristics.

(1) The set is ordered: $s^{(g)}_{f(i)} \geq s^{(g)}_{f(j)}$ if $i \geq j$.

(2) $s^{(g)}_{f(\frac{g-1}{2})}$ is the central term.

(3) There is a negation operator: $neg(s^{(g)}_{f(i)}) = s^{(g)}_{f(j)}$ such that $j = (g-1) - i$.

(4) Max operator: $\max(s^{(g)}_{f(i)}, s^{(g)}_{f(j)}) = s^{(g)}_{f(i)}$, if $s^{(g)}_{f(i)} \geq s^{(g)}_{f(j)}$.

(5) Min operator: $\min(s^{(g)}_{f(i)}, s^{(g)}_{f(j)}) = s^{(g)}_{f(i)}$, if $s^{(g)}_{f(i)} \leq s^{(g)}_{f(j)}$.

In what follows, the balanced and unbalanced linguistic sets are defined by comparing the spacings between the numerical scales of adjacent linguistic terms.

Definition 3. Let $S^f_g = \{s_{f(0)}, s_{f(1)}, \ldots, s_{f(i)}, \ldots, s_{f(g-1)}\}$ be a scaled linguistic set. If $f(i+1) - f(i) = f(i+2) - f(i+1)$ $(i = 0, 1, \ldots, g-3)$, then S^f_g is called a balanced linguistic term set; otherwise, an unbalanced linguistic set.

Example 1. Let $f_1(x) = \lfloor x \rfloor$ ($\lfloor \cdot \rfloor$ represents rounding down operator), $f_2(x) = \sqrt{x}$ and $f_3(x) = x\sqrt{x}$. Then we obtain $S^{f_1}_5 = \{s_0, s_1, s_2, s_3, s_4\}$, $S^{f_2}_5 = \{s_0, s_1, s_{1.41}, s_{1.73}, s_2\}$ and $S^{f_3}_5 = \{s_0, s_1, s_{2.89}, s_{5.19}, s_8\}$, respectively. $S^{f_1}_5$ is a balanced linguistic set. Both $S^{f_2}_5$ and $S^{f_3}_5$ are unbalanced linguistic sets.

Example 1 indicates that different linguistic term sets with various semantics can be constructed by different scaling functions. Note that different scaling functions could also determine the linguistic term set with the same semantics. $f(x) = x$ and $f_1(x) = \lfloor x \rfloor$ are the cases in point. Next, we will show that the function with special properties can scale the linguistic term sets with different characteristics.

Proposition 1. *Let $S^f_g = \{s_{f(0)}, s_{f(1)}, \ldots, s_{f(i)}, \ldots, s_{f(g-1)}\}$ be a linguistic set scaled by f. f is strictly monotonic increasing, continuous on $[0, g-1]$ and twice differentiable on $(0, g-1)$.*

(1) *If $f''(x) = 0$, then S^f_g is a balanced linguistic set.*

(2) *If $f''(x) > 0$, then S^f_g is an unbalanced linguistic set with $f(j+1) - f(j) < f(i+1) - f(i)$ $(0 \leq j < i < g-1)$.*

(3) *if $f''(x) < 0$, then S^f_g is an unbalanced linguistic set with $f(j+1) - f(j) > f(i+1) - f(i)$ $(0 \leq j < i < g-1)$.*

Proof. (1) Since f is strictly monotonic increasing, twice differentiable on $[0, g-1]$, $f''(x) = 0$ means $f(x) = a \cdot x + b$ $(a > 0)$. $f(i+1) - f(i) = a(i+1) + b - (ai + b) = a = a(i+2) + b - (a(i+1) + b) = f(i+2) - f(i+1)$. Therefore, S^f_g is a balanced linguistic set.

(2) Since $f(x)$ is continuous on $[0, g-1]$, and twice differentiable on $(0, g-1)$, it is differentiable on $(0, g-1)$. According to Lagrange Mean Value Theorem, there exist $\xi_1 \in (j, j+1)$, $\xi_2 \in (i, i+1)$, such that $f(j+1) - f(j) = f'(\xi_1) \times (j+1-j) = f'(\xi_1)$ and $f(i+1) - f(i) = f'(\xi_2) \times (i+1-i) = f'(\xi_2)$. From the conditions that $f''(x) > 0$ and $j < \xi_1 < j+1 \leq i < \xi_2 < i+1$, $f'(\xi_1) < f'(\xi_2)$ holds. Hence, $f(j+1) - f(j) < f(i+1) - f(i)$. Therefore, S^f_g is an unbalanced linguistic set.

(3) Follows in the similar manner as (2).

We can draw these conclusions from Proposition 1. If the scaling function is *convex*, then S_g^f has the characteristic that the further a linguistic term from the first term $s_{f(0)}$, the larger the spacing between the numerical scales of the term and its adjoining linguistic term; if the scaling function is *concave*, then S_g^f has the characteristic that the further, the smaller.

We now use Proposition 1 to analyze $S_5^{f_2}$ and $S_5^{f_3}$ in Example 1. Since $f_2(x) = \sqrt{x}$, $f_3(x) = x\sqrt{x}$, $f_2''(x) = -\frac{1}{4x\sqrt{x}} < 0$ and $f_3''(x) = \frac{3}{4\sqrt{x}} > 0$ on $(0, 4)$, then the two linguistic term sets are unbalanced and with the characteristics given above.

Example 2. Xu[25] defined a family of unbalanced linguistic sets as follows and applied them in a practical multi-criteria decision making problem successfully.

$$S'_{2t-1} = \left\{ s_i^{(2t-1)} \mid i = -\frac{2(t-1)}{t+2-t}, -\frac{2(t-1-1)}{t+2-(t-1)}, \ldots, -\frac{2\times(2-1)}{t+2-2}, \frac{2\times(1-1)}{t+2-1}, \frac{2(2-1)}{t+2-2}, \ldots, \frac{2(t-1-1)}{t+2-(t-1)}, \frac{2(t-1)}{t+2-t} \right\} \quad (t = 2, 3, \ldots, n).$$

In fact, the unbalanced linguistic set S'_{2t-1} is equivalent to S_{2t-1}^f in which $f(x) = \frac{2(x-t+1)}{t+1-|x-t+1|}$ $(0 \le x \le 2(t-1))$. The characteristic of numerical scales can also be explained by Proposition 1. $f(x) = \frac{2(x-t+1)}{t+1-|x-t+1|}$ is continuous on $[0, 2t-2]$. If $0 \le x \le (t-1)$, $f(x) = \frac{2(x-t+1)}{t+1-|x-t+1|} = \frac{2(x-t+1)}{2+x}$, $f'(x) = \frac{2(t+1)}{(2+x)^2} > 0$ and $f''(x) = \frac{-4(t+1)}{2+x}^3 < 0$. If $(t-1) < x \le 2(t-1)$, $f(x) = \frac{2(x-t+1)}{t+1-|x-t+1|} = \frac{2(x-t+1)}{2t-x}$, $f'(x) = \frac{2(t+1)}{(2t-x)^2} > 0$ and $f''(x) = \frac{4(t+1)}{2t-x}^3 > 0$. According to Proposition 1, S_{2t-1}^f is an unbalanced linguistic set and the linguistic terms in it distribute unbalancedly with the characteristic that the further a linguistic term from the central term, the greater the absolute value of the deviation between the scales of this term and its adjoining term.

In the remainder of this subsection, we will extend the classical 2-tuple linguistic model into a new one based on the scaled linguistic set S_g^f.

Definition 4. Let $S_g^f = \{s_{f(0)}, s_{f(1)}, \ldots, s_{f(i)}, \ldots, s_{f(g-1)}\}$ be a linguistic set and $\beta \in [f(0), f(g-1)]$ be a value representing the result of a symbolic aggregation operation, then the 2-tuple linguistic that expresses the equivalent information to β is obtained from the following function $\widetilde{\Delta}_g^f : [f(0), f(g-1)] \to S_g^f \times [-50\%, 50\%)$,

$$\widetilde{\Delta}_g^f(\beta) = (s_{f(k)}, 100\alpha\%) \text{ with} \begin{cases} f(i) \le \beta < f(i+1), \\ \alpha = \overline{sgn}\left(\frac{f(i+1)-f(i)}{2} - \beta\right) \times min\left(\frac{\beta - f(i)}{f(i+1)-f(i)}, \frac{f(i+1)-\beta}{f(i+1)-f(i)}\right), \end{cases} \tag{2}$$

where $k = \frac{1+sgn(\alpha)}{2} f(i) + \frac{1-sgn(\alpha)}{2} f(i+1)$, $sgn(x)$ is the usual sign function, and $\overline{sgn}(x) = \begin{cases} 1, & x > 0, \\ -1, & x \le 0. \end{cases}$

In the above definition, $f(g) = f(g-1)$ is set to be compatible with Eq. (2).

Apparently, $100\alpha\%$ represents **the ratio of the difference**, which is between β and the closet scaled terms $s_{f(k)}$, to the length of the scaled interval that β falls into.

Example 3. Suppose $\beta = 1.83$ is obtained from a symbolic aggregation operation over the terms in $S_5^{\sqrt{x}} = \{s_0, s_1, s_{1.41}, s_{1.73}, s_2\}$. Then, the value can be represented by means of a new 2-tuple linguistic as $\widetilde{\Delta}_5^{\sqrt{3}}(1.83) = (s_{1.73}, 100 \times 0.37\%) = (s_{\sqrt{3}}, 37\%)$. Since $1.73 < 1.83 \leq 2$, $1.83 < \frac{1.73+2}{2} \approx 1.87$ and $\frac{1.83-1.73}{2-1.73} = 0.37 < 0.63 = \frac{2-1.83}{2-1.73}$, $\alpha = \overline{sgn}(1.87 - 1.83) \times 0.37 = 0.37$.

Let $S_g^f = \{s_{f(0)}, s_{f(1)}, \ldots, s_{f(i)}, \ldots, s_{f(g-1)}\}$ be a linguistic set and $(s_{f(i)}, 100\alpha\%)$ be a new 2-tuple linguistic. Denote $\overline{S_g^f} = (S_g^f - \{s_{f(0), s_{f(g-1)}}\}) \times [-50\%, 50\%) \cup \{s_{f(0)}\} \times [0\%, 50\%) \cup \{s_{f(g-1)}\} \times [-50\%, 0\%]$. The function $(\widetilde{\Delta}_g^f)^{-1}: \overline{S_g^f} \to [f(0), f(g-1)]$ which returns the equivalent numerical value $\beta \in [f(0), f(g-1)]$ from a 2-tuple linguistic is defined as

$$(\widetilde{\Delta}_g^f)^{-1}(s_{f(i)}, 100\alpha\%) = \beta = \begin{cases} f(i) + \alpha \cdot (f(i+1) - f(i)), & \alpha \geq 0\,, \\ f(i) + \alpha \cdot (f(i) - f(i-1)), & \alpha < 0\,. \end{cases} \tag{3}$$

It is obvious that the conversion of a new linguistic term in S_g^f into a 2-tuple linguistic consists of adding a value 0% as symbolic translation: $s_{f(i)}^{(g)} \in S_g^f \Rightarrow (s_{f(i)}^{(g)}, 0\%)$. The new 2-tuple linguistic is the same as Dong's model. However, Dong[27] gave the mapping form 2-tuple to real numbers in R and did not define the reverse mapping. This study perfects Dong's 2-tuple representation model. The following is the computational model for the new extended 2-tuple linguistic representation.

(1) *Comparison operators of the new 2-tuple linguistic*:

The comparison of linguistic information represented by the new 2-tuples can also be carried out according to an ordinary lexicographic order. Let $(s_{f(k)}^{(g)}, 100\alpha_1\%)$ and $(s_{f(l)}^{(g)}, 100\alpha_2\%)$ be two 2-tuples, with each one representing a counting of information as follows:

(i) if $k < l$, then $(s_{f(k)}^{(g)}, 100\alpha_1\%)$ is smaller than $(s_{f(l)}^{(g)}, 100\alpha_2\%)$;

(ii) if $k = l$, then

(a) if $\alpha_1 = \alpha_2$ then $(s_{f(k)}^{(g)}, 100\alpha_1\%)$ and $(s_{f(l)}^{(g)}, 100\alpha_2\%)$ represent the same information;

(b) if $\alpha_1 < \alpha_2$ then $(s_{f(k)}^{(g)}, 100\alpha_1\%)$ is smaller than $(s_{f(l)}^{(g)}, 100\alpha_2\%)$;

(c) if $\alpha_1 > \alpha_2$ then $(s_{f(k)}^{(g)}, 100\alpha_1\%)$ is bigger than $(s_{f(l)}^{(g)}, 100\alpha_2\%)$.

(2) *Negation operator of the new 2-tuple linguistic:*

$$Neg((s^{(g)}_{f(i)}, 100\alpha\%)) = \begin{cases} (s^{(g)}_{f(g-i)}, 100\alpha\%), & \alpha = -0.5\,, \\ (s^{(g)}_{f(g-1-i)}, -100\alpha\%), & -0.5 \neq \alpha < 0.5\,. \end{cases}$$

(3) *Aggregation operators of the new 2-tuples*:

All the aggregation operators presented in Ref. 2 can be adjusted for the extended 2-tuples. We take the weighted average operator for examples, Let $\overrightarrow{x} = ((s^{(g)}_{f(k_1)}, 100\alpha_1\%), (s^{(g)}_{f(k_2)}, 100\alpha_2\%), \dots, (s^{(g)}_{f(k_n)}, 100\alpha_n\%))$ be a vector of the new 2-tuples and $\overrightarrow{\omega} = (\omega_1, \omega_2, \dots, \omega_n)$ be an associated weighting vector that satisfies $w_i \in [0,1]$ and $\sum_{i=1}^n \omega_i = 1$. The new 2-tuple weighted average (WA) $\overline{x}^{\overrightarrow{\omega}}$ is computed as

$$\overline{x}^{\overrightarrow{\omega}} = \widetilde{\Delta}^f_g \left(\sum_{i=1}^n \omega_i (\widetilde{\Delta}^f_g)^{-1} (s^{(g)}_{f(k_i)}, 100\alpha_i\%) \right). \tag{4}$$

Other operators can also be adjusted by substituting $\widetilde{\Delta}^f_g$ for Δ as the way shown above.

The scale domains of the multi-granular linguistic sets are quite different even with the same scaling function. Thus the transformation process is needed to unify the multi-granular linguistic sets in the literature. While, executing the transformation process would increase the complexity of the computation, and the choice of the transformation functions also bring the subjectivity in decision making. In order to avoid the transformation process, we are going to use a kind of special scaling functions to attain the semantics of the multi-granular linguistic sets exactly in the unit interval in the following subsection.

3.2. *A normalized numerical scaling method*

This subsection presents a normalized method for scaling the multi-granular linguistic sets into a single interval [0,1].

Definition 5.[45] Let $\mathfrak{f}: [0,1] \to [0,1]$ be a function and have the properties that $\mathfrak{f}(0) = 0$, $\mathfrak{f}(1) = 1$ and $\mathfrak{f}(x) \geq \mathfrak{f}(y)$ if $x > y$. Then the function is called a basic unit-interval monotonic (BUM) function.

The BUM function is monotonic and fixed at the end points. If $a > 0$, $\mathfrak{f}(x) = x^a$ $(x \in [0,1])$ is a BUM function. $\mathfrak{f}(x) = log_2(x+1)$ $(x \in [0,1])$ is also a BUM function. However, $f(x) = x^a$ $(x \in R)$ and $f(x) = log_2(x+1)$ $(x \in (0,+\infty))$ are not BUM functions, since the domains of these two functions are not on [0,1].

Definition 6. Let $\mathfrak{f}(x)$ be a BUM function and $h_g(x) = \frac{x}{g-1}$ $(g > 1)$. Let $F_g(x) = \mathfrak{f} \circ h_g(x) = \mathfrak{f}(\frac{x}{g-1})$. Then,

$$\mathcal{S}^{F_g}_g = \{s^{(g)}_{F_g(0)}, s^{(g)}_{F_g(1)}, \dots, s^{(g)}_{F_g(i)}, \dots, s^{(g)}_{F_g(g-1)}\}$$

is a normalized linguistic set scaled by F_g and determined by $\mathfrak{f}$, where $F_g(i) = \mathfrak{f}(\frac{i}{g-1})$ is the normalized numerical scale of the ith term.

The BUM function $\mathfrak{f}(x)$ is supposed to be strictly monotonic in most cases. $F_g(x)$ $(x \in [0, g-1])$ is the normalized numerical scaling function for the linguistic set with the granularity of g, and we denote $\mathcal{S}_g^{F_g}$ as $\mathcal{S}_g^F$ for convenience.

Remark 1. The granularity of a linguistic set is g $(g \geq 3)$, which means a scaling function is defined on $[0, g]$. This fact implies that a single BUM function $\mathfrak{f}(x)$ with the domain of [0,1] can not scale a linguistic set directly. Therefore, one needs to combine a BUM function with $h_g(x)$ to scale a linguistic set into a normalized one.

Let $\mathfrak{f}(x) = b(x)$ $(x \in [0,1])$ be a BUM function and $f(x) = b(x)$ $(x \in [0, g-1])$ be a new function extended from $\mathfrak{f}(x)$. Then S_g^f and $\mathcal{S}_g^F = \mathcal{S}_g^{\mathfrak{f} \circ h_g} = S_g^{b \circ h_g}$ are two different linguistic sets for their different scaling functions. Let $\mathfrak{f}_1(x) = x^2$ $(x \in [0,1])$ and $f_1(x) = x^2$ $(x \in [0,2])$ be a function extended from $\mathfrak{f}_1(x)$. $S_3^{f_1} = \{s_0, s_1, s_{1.41}\}$ and $S_3^{F_3} = S_3^{\mathfrak{f}_1 \circ h_3} = \{s_0, s_{0.25}, s_1\}$. The former is a general linguistic set and the latter is a normalized linguistic set.

Since the numerical scales of multi-granular linguistic sets in Definition 6 are restricted on $[0,1]$, then we need not transform the terms from one granularity to another.

Example 4. $\mathfrak{f}(x) = \sqrt{x}$ $(x \in [0,1])$ us a BUM function. The normalized linguistic set scaled by $F_3(x)$ with the granularity of three is calculated as $\mathcal{S}_3^F = \mathcal{S}_3^{F_3} = \{s^{(3)}_{\sqrt{\frac{i}{3-1}}} \mid i = 0,1,2\} = \{s^{(3)}_{\sqrt{\frac{0}{3-1}}}, s^{(3)}_{\sqrt{\frac{1}{3-1}}}, s^{(3)}_{\sqrt{\frac{2}{3-1}}}\} = \{s_0^{(3)}, s_{0.71}^{(3)}, s_1^{(3)}\}$. Analogously, $\mathcal{S}_5^F = \{s_0^{(5)}, s_{0.50}^{(5)}, s_{0.71}^{(5)}, s_{0.86}^{(5)}, s_1^{(5)}\}$ and $\mathcal{S}_7^F = \{s_0^{(7)}, s_{0.41}^{(7)}, s_{0.58}^{(7)}, s_{0.71}^{(7)}, s_{0.82}^{(7)}, s_{0.91}^{(7)}, s_1^{(7)}\}$.

Proposition 2. *If* $\mathfrak{f}(x) = x^a$ $(a > 0, a \neq 1, 0 \leq x \leq 1)$, *then* $\mathcal{S}_g^F$ *is a normalized and unbalanced linguistic term set. Suppose* i *and* j *are two non-negative integers and* $0 \leq j < i < g-1$. *Then,*

(1) *if* $0 < a < 1$, *then* $F_g(j+1) - F_g(j) > F_g(i+1) - F_g(i)$, *i.e.* $\mathfrak{f}(\frac{j+1}{g-1}) - \mathfrak{f}(\frac{j}{g-1}) > \mathfrak{f}(\frac{i+1}{g-1}) - \mathfrak{f}(\frac{i}{g-1})$;
(2) *if* $a > 1$, *then* $F_g(j+1) - F_g(j) < F_g(i+1) - F_g(i)$, *i.e.* $\mathfrak{f}(\frac{j+1}{g-1}) - \mathfrak{f}(\frac{j}{g-1}) < \mathfrak{f}(\frac{i+1}{g-1}) - \mathfrak{f}(\frac{i}{g-1})$.

Proof. It is easy to verify that $\mathfrak{f}(x) = x^a$ $(a > 0, a \neq 1, 0 \leq x \leq 1)$ is a BUM function, so $\mathcal{S}_g^F$ is a normal linguistic set. Since $F_g(x) = \mathfrak{f}(\frac{x}{g-1}) = (\frac{x}{g-1})^a$ $(a > 0, a \neq 1)$ is continuous on $[0, g-1]$, and differentiable in $(0, g-1)$. According to Lagrange Mean Value Theorem, there exist $\xi_1 \in (0,1)$ and $\xi_2 \in (1,2)$ such that $F_g(1) - F_g(0) = F_g'(\xi_1)(1-0) = \frac{1}{g-1}\mathfrak{f}'(\frac{\xi_1}{g-1})$ and $F_g(2) - F_g(1) = F_g'(\xi_2)(2-1) = \frac{1}{g-1}\mathfrak{f}'(\frac{\xi_2}{g-1})$. Since $\xi_1 \neq \xi_2$, $F_g(1) - F_g(0) \neq F_g(2) - F_g(1)$. Therefore, $\mathcal{S}_g^F$ is a normalized and unbalanced linguistic set.

The proofs of the two items follow in a similar manner as Proposition 1, so they are omitted here.

Table 1. Teachers' linguistic evaluations.

	c_1	c_2	c_3	c_4	c_5
a_1	f	g	vg	vp	p
a_2	g	f	g	p	p
a_3	f	vg	eg	ep	vp

Table 2. The linguistic evaluations in $\overline{\mathcal{S}_7^F}$.

	c_1	c_2	c_3	c_4	c_5	WA result
a_1	$(s_{0.50}, 0\%)$	$(s_{0.67}, 0\%)$	$(s_{0.83}, 0\%)$	$(s_{0.17}, 0\%)$	$(s_{0.33}, 0\%)$	$(s_{F_7(3)}, 0\%)$
a_2	$(s_{0.67}, 0\%)$	$(s_{0.50}, 0\%)$	$(s_{0.67}, 0\%)$	$(s_{0.33}, 0\%)$	$(s_{0.33}, 0\%)$	$(s_{F_7(3)}, 0\%)$
a_3	$(s_{0.50}, 0\%)$	$(s_{0.83}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.00}, 0\%)$	$(s_{0.17}, 0\%)$	$(s_{F_7(3)}, 0\%)$

The following example illustrates that different scaling functions can yield completely different results.

Example 5. Consider a multi-criteria decision making problem. Three students a_1, a_2 and a_3 have taken an examination which covers five equal weighted subjects c_1, c_2, c_3, c_4 and c_5. Their teachers have evaluated the students by using the linguistic set $S_7 = \{s_0 : \textit{extremely poor } (ep),\ s_1 : \textit{very poor } (vp),\ s_2 : \textit{poor } (p),\ s_3 : \textit{fair } (f),\ s_4 : \textit{good } (g),\ s_5 : \textit{very good } (vg),\ s_6 : \textit{etremely good } (eg)\}$ (See Table 1).

Suppose that the above linguistic set is scaled by three different functions $F_7(x) = \frac{x}{7-1}$, $\Phi_7(x) = (\frac{x}{7-1})^{1.1}$ and $\Psi_7(x) = (\frac{x}{7-1})^{0.9}$ (the corresponding BUM functions are $\mathfrak{f}(x) = x$ $(x \in [0,1])$, $'(x) = x^{1.1}$ $(x \in [0,1])$, and $(x) = x^{0.9}$ $(x \in [0,1])$). Then, S_7 is computed with different scales as: $\mathcal{S}_7^F = \{s_0, s_{0.17}, s_{0.33}, s_{0.50}, s_{0.67}, s_{0.83}, s_1\}$, $\mathcal{S}_7^\Phi = \{s_0, s_{0.14}, s_{0.30}, s_{0.47}, s_{0.64}, s_{0.82}, s_1\}$ and $\mathcal{S}_7^\Psi = \{s_0, s_{0.20}, s_{0.37}, s_{0.54}, s_{0.69}, s_{0.85}, s_1\}$. The scaled linguistic evaluations are shown in Tables 2–4.

Next, we calculate the comprehensive results of the students by using Eq. (4) and show the WA results in the last column of Tables 2–4. Take the aggregation result of a_1 in $\overline{\mathcal{S}_7^\Phi}$ for example,

$$\begin{aligned}
&\widetilde{\Delta}_7^\Phi(\tfrac{1}{5} \times ((\widetilde{\Delta}_7^\Phi)^{-1}(s_{0.47}, 0\%) + (\widetilde{\Delta}_7^\Phi)^{-1}(s_{0.64}, 0\%) \\
&\qquad + (\widetilde{\Delta}_7^\Phi)^{-1}(s_{0.82}, 0\%) + (\widetilde{\Delta}_7^\Phi)^{-1}(s_{0.14}, 0\%) + (\widetilde{\Delta}_7^\Phi)^{-1}(s_{0.30}, 0\%))) \\
&\quad = \widetilde{\Delta}_7^\Phi(\tfrac{1}{5} \times (0.47 + 0.64 + 0.82 + 0.14 + 0.33)) \\
&\quad = \widetilde{\Delta}_7^\Phi(0.474) \\
&\quad = (s_{0.47}, 2\%) \\
&\quad = (s_{\Phi_7(3)}, 2\%)\,.
\end{aligned}$$

In the first case, with the term set $\overline{\mathcal{S}_7^F}$, the comprehensive performances of the three students are all equal to $(s_{0.50}, 0\%)$ (See Table 2); in the second case, with the term set $\overline{\mathcal{S}_7^\Phi}$, a_3 ranks first, followed by a_1 and then a_2 (See Table 3); in the

Table 3. The linguistic evaluations in $\overline{\mathcal{S}_7^{\Phi}}$.

	c_1	c_2	c_3	c_4	c_5	WA result
a_1	$(s_{0.47}, 0\%)$	$(s_{0.64}, 0\%)$	$(s_{0.82}, 0\%)$	$(s_{0.14}, 0\%)$	$(s_{0.30}, 0\%)$	$(s_{\Phi_7(3)}, 2\%)$
a_2	$(s_{0.64}, 0\%)$	$(s_{0.47}, 0\%)$	$(s_{0.64}, 0\%)$	$(s_{0.30}, 0\%)$	$(s_{0.30}, 0\%)$	$(s_{\Phi_7(3)}, 0\%)$
a_3	$(s_{0.47}, 0\%)$	$(s_{0.82}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.00}, 0\%)$	$(s_{0.14}, 0\%)$	$(s_{\Phi_7(3)}, 9\%)$

Table 4. The linguistic evaluations in $\overline{\mathcal{S}_7^{\Psi}}$.

	c_1	c_2	c_3	c_4	c_5	WA result
a_1	$(s_{0.54}, 0\%)$	$(s_{0.69}, 0\%)$	$(s_{0.85}, 0\%)$	$(s_{0.20}, 0\%)$	$(s_{0.37}, 0\%)$	$(s_{\Psi_7(3)}, -6\%)$
a_2	$(s_{0.69}, 0\%)$	$(s_{0.54}, 0\%)$	$(s_{0.69}, 0\%)$	$(s_{0.37}, 0\%)$	$(s_{0.37}, 0\%)$	$(s_{\Psi_7(3)}, -5\%)$
a_3	$(s_{0.54}, 0\%)$	$(s_{0.85}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.00}, 0\%)$	$(s_{0.20}, 0\%)$	$(s_{\Psi_7(3)}, -13\%)$

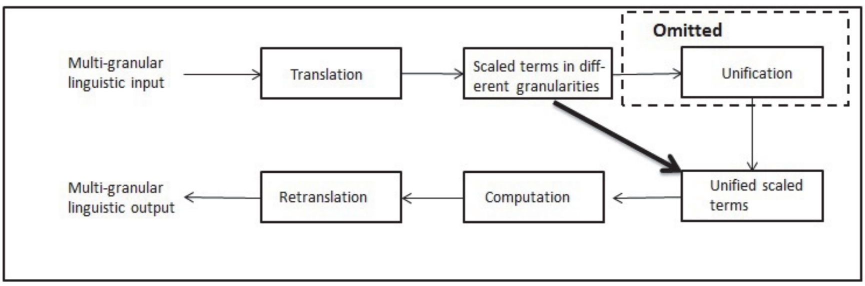

Fig. 3. The computation scheme of the normalized multi-granular linguistic terms.

last case, with the term set $\overline{\mathcal{S}_7^{\Psi}}$, a_2 ranks first, followed by a_1 and then a_3 (See Table 4).

We further summarize the computation scheme of the proposed normalized multi-granular linguistic terms in Fig. 3. The dashed line represents that one characteristic of our method's is to omit the unification processes.

As shown in Example 5, different scaling functions might lead to entirely different decision results. Therefore, obtaining a suitable scaling function is a very important task. In the next section, we will construct two classes of normalized scaling functions with consideration of the priori scale information.

4. Two Interpolant Methods for Constructing Normalized Scaling Functions

As analyzed in Introduction, the numerical scale of a linguistic set S_g can be provided by a decision maker or calculated by some existing methods,[1,42,43] and the numerical scales of the linguistic sets with other granularities are unknown. The unknown scales make it impossible to execute the consensus process or the aggregation process. Thus, it is a vital task to estimate the numerical scales of the other linguistic sets. In this section, considering the priori scale information, two kinds of piecewise interpolation scaling functions are constructed to estimate the scales of the multi-granular linguistic sets.

Let $S_g = \{s_{NS(0)}, s_{NS(1)}, \ldots, s_{NS(i)}, \ldots, s_{NS(g-1)}\}$ be a scaled linguistic set (the priori scaled linguistic set). For the sake of convenience, the priori scale information is denoted as a point set $\{(0, NS(0)), (1, NS(1)), \ldots, (i, NS(i)), \ldots, (g-1, NS(g-1))\}$, where $0 = NS(0) \leq NS(i) < NS(i+1) \leq NS(g-1) = 1$.

Apparently, these g points are distinct. Our objective is to find a relatively smooth and easily calculated BUM function $\mathfrak{f}(x)$ such that $F_g(i) = \mathfrak{f} \circ h_g(i) = \mathfrak{f}(\frac{i}{g-1}) = NS(i)$ $(i = 0, 1, \ldots, g-1)$. Therefore, the problem is converted to a general interpolation problem as follows.

Given a set of scaled points

$$\mathcal{A} = \left\{\left(\frac{0}{g-1}, 0\right), \left(\frac{1}{g-1}, NS(1)\right), \ldots, \left(\frac{i}{g-1}, NS(i)\right), \ldots, \left(\frac{g-1}{g-1}, 1\right)\right\}, \quad (5)$$

we interpolate a BUM function $\widehat{\mathfrak{f}}(x)$ such that $\widehat{\mathfrak{f}}(\frac{i}{g-1}) = NS(i)$ $(i = 0, 1, \ldots, g-1)$.

Let $S_{g'}$ be a linguistic set. Then

$$\mathcal{S}_{g'}^{\widehat{F}} = \mathcal{S}_{g'}^{\widehat{F}_{g'}} = \left\{s_{\widehat{\mathfrak{f}}(\frac{0}{g'-1})}, s_{\widehat{\mathfrak{f}}(\frac{1}{g'-1})}, \ldots, s_{\widehat{\mathfrak{f}}(\frac{i}{g'-1})}, \ldots, s_{\widehat{\mathfrak{f}}(\frac{g'-1}{g'-1})}\right\}$$

is the normalized scaled linguistic set determined by the interpolation function $\widehat{\mathfrak{f}}(x)$.

4.1. *Piecewise linear interpolation*

Given the scaled points set $\mathcal{A}$ in Eq. (5). The piecewise linear interpolation function is constructed as

$$\widehat{l}(x) = \begin{cases} NS(0) + (g-1)\cdot(NS(1) - NS(0))\,(x - \frac{0}{g-1}), & 0 \leq x < \frac{1}{g-1}, \\ \cdots & \cdots \\ NS(i) + (g-1)\cdot(NS(i+1) - NS(i))(x - \frac{i}{g-1}), & \frac{i}{g-1} \leq x < \frac{i+1}{g-1}, \\ \cdots & \cdots \\ NS(g-2) + (g-1)\cdot(NS(g-1) - NS(g-2))(x - \frac{g-2}{g-1}), & \frac{g-2}{g-1} \leq x \leq 1. \end{cases} \quad (6)$$

The piecewise linear interpolation function $\widehat{l}(x)$ is shown in Fig. 4, and the linguistic set with a finer granularity determined by $\widehat{l}(x)$ may be local balanced between the two adjacent priori linguistic terms.

Example 6. Suppose $\mathcal{S}_5 = \{s_0, s_{0.24}, s_{0.50}, s_{0.76}, s_1\}$ is a priori scaled term set. The interpolation function $\widehat{l}(x)$ is computed as

$$\widehat{l}(x) = \begin{cases} 0 + 0.96 \times (x - 0), & 0 \leq x < 0.25, \\ 0.24 + 1.04 \times (x - 0.25), & 0.25 \leq x < 0.5, \\ 0.50 + 1.04 \times (x - 0.50), & 0.50 \leq x < 0.75, \\ 0.76 + 0.96 \times (x - 0.75), & 0.75 \leq x \leq 1. \end{cases}$$

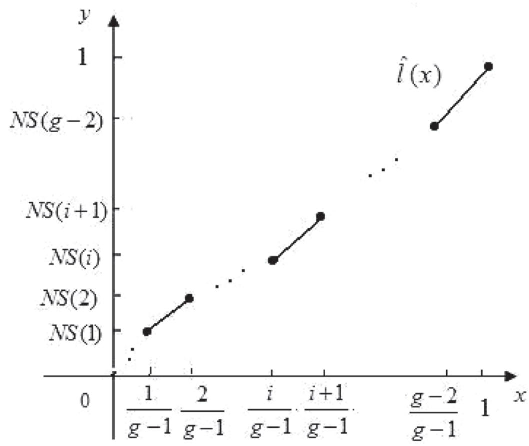

Fig. 4. Piecewise linear interpolation function.

Using the scaling function $\widehat{L}_7(x) = \widehat{l}(\frac{x}{7-1})$, the linguistic set with seven granularity can be scaled as

$$\mathcal{S}_7^{\widehat{L}} = \mathcal{S}_7^{\widehat{L}_7} = \{s_{\widehat{l}(\frac{i}{7-1})} \mid i = 0, 1, \ldots, 6\} = \{s_{\mathbf{0}}, s_{0.16}, s_{0.33}, s_{\mathbf{0.50}}, s_{0.67}, s_{0.84}, s_{\mathbf{1}}\}.$$

Similarly, we have $\mathcal{S}_3^{\widehat{L}} = \{s_{\mathbf{0}}, s_{\mathbf{0.50}}, s_{\mathbf{1}}\}$ and $\mathcal{S}_9^{\widehat{L}} = \{s_{\mathbf{0}}, s_{0.12}, s_{\mathbf{0.24}}, s_{0.37}, s_{\mathbf{0.50}}, s_{0.63}, s_{\mathbf{0.76}}, s_{0.88}, s_{\mathbf{1}}\}$.

The numerical scales in bold show that the numerical scales of the linguistic term sets with other granularities include the priori numerical scale information as much as possible.

Interestingly, if $(g-1)|(g'-1)$, then parts of numerical scales in $\mathcal{S}_{g'}^{\widehat{L}}$ are the same as the priori scales; and if $(g'-1)|(g-1)$, then all of the numerical scales in $\mathcal{S}_{g'}^{\widehat{L}}$ are the same as the priori scales.

4.2. *Piecewise circular arc interpolation*

Given the scaled points set $\mathcal{A}$ in Eq. (5) and divided it into $\frac{g-1}{2}$ groups of adjoining points $\{(\frac{2i}{g-1}, NS(2i)), (\frac{2i+1}{g-1}, NS(2i+1)), (\frac{2i+2}{g-1}, NS(2i+2))\}$ $(i = 0, \ldots, \frac{g-3}{2})$. Then the piecewise circular arc function $\widehat{c}(x)$ is interpolated as

$$\widehat{c}(x) = \begin{cases} c_0(x), & 0 \le x < \frac{2}{g-1}, \\ \cdots & \cdots \\ c_i(x), & \frac{2i}{g-1} \le x < \frac{2i+2}{g-1}, \\ \cdots & \cdots \\ c_{\frac{g-3}{2}}(x), & \frac{g-3}{g-1} \le x \le 1, \end{cases} \tag{7}$$

where $y = c_i(x)$ is in the form of implicit function

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ (\frac{2i}{g-1})^2 + (NS(2i))^2 & \frac{2i}{g-1} & NS(2i) & 1 \\ (\frac{2i+1}{g-1})^2 + (NS(2i+1))^2 & \frac{2i+1}{g-1} & NS(2i+1) & 1 \\ (\frac{2i+2}{g-1})^2 + (NS(2i+2))^2 & \frac{2i+2}{g-1} & NS(2i+2) & 1 \end{vmatrix} = 0 \tag{8}$$

demanding that $NS(2i) \leq y \leq NS(2i+2)$.

The implicit function is the equation of the circle passing the three points $(\frac{2i}{g-1}, NS(2i)), (\frac{2i+1}{g-1}, NS(2i+1))$ and $(\frac{2i+2}{g}, NS(2i+2))$. The condition $NS(2i) \leq y \leq NS(2i+2)$ ensures that the solution of Eq. (8) is exactly what we need. The piecewise circle arc interpolation function $\widehat{c}(x)$ is shown in Fig. 5.

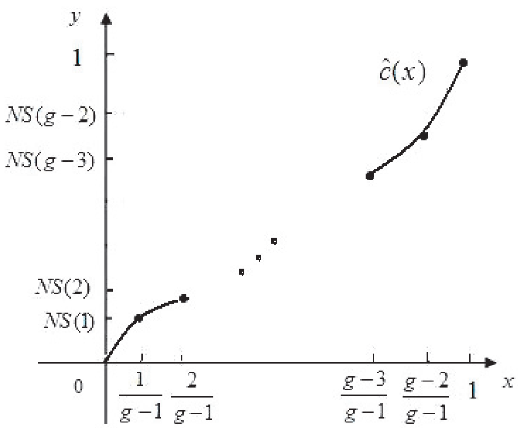

Fig. 5. Piecewise circular arc interpolation function.

Example 7. The priori information $\mathcal{S}_5 = \{s_0, s_{0.24}, s_{0.5}, s_{0.76}, s_1\}$ is used again to interpolate the piecewise circle arc scaling function $\widehat{C}_g(x)$. First, the BUM function $\widehat{c}(x)$ should be calculated. Since $\frac{g-1}{2} = \frac{5-1}{2} = 2$, the BUM function is constructed by two adjoining circular arcs.

$$\widehat{c}(x) = \begin{cases} c_0(x), & 0 \leq x < \frac{2}{5-1}, \\ c_1(x), & \frac{2}{5-1} \leq x \leq 1, \end{cases}$$

where $c_0(x)$ and $c_1(x)$ are

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 0 & 0 & 0 & 1 \\ 0.25^2+0.24^2 & 0.25 & 0.24 & 1 \\ 0.50^2+0.50^2 & 0.50 & 0.50 & 1 \end{vmatrix} = 0 \quad (0 \leq y \leq 0.50)$$

and

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 0.50^2+0.50^2 & 0.50 & 0.50 & 1 \\ 0.75^2+0.76^2 & 0.75 & 0.76 & 1 \\ 1^2+1^2 & 1 & 1 & 1 \end{vmatrix} = 0 \quad (0.50 \le y \le 1),$$

respectively.

The linguistic sets $\mathcal{S}_3^{\widehat{C}}$, $\mathcal{S}_7^{\widehat{C}}$ and $\mathcal{S}_9^{\widehat{C}}$ are calculated from $\widehat{C}_g(x) = \widehat{c}(\frac{x}{g-1})$ $(g = 3, 7, 9)$. $S_3^{\widehat{C}} = \{\boldsymbol{s_0}, \boldsymbol{s_{0.50}}, \boldsymbol{s_1}\}$, $S_7^{\widehat{C}} = \{\boldsymbol{s_0}, s_{0.15}, s_{0.33}, \boldsymbol{s_{0.50}}, s_{0.68}, s_{0.84}, \boldsymbol{s_1}\}$, $S_9^{\widehat{C}} = \{\boldsymbol{s_0}, s_{0.12}, \boldsymbol{s_{0.24}}, s_{0.35}, \boldsymbol{s_{0.50}}, s_{0.64}, \boldsymbol{s_{0.76}}, s_{0.95}, \boldsymbol{s_1}\}$. The meaning of the numerical scales in bold is the same as that in Example 6.

5. Comparison the New Linguistic Representation Method with Several Representative Methods

In this section, we compare the normalized multi-granular unbalanced linguistic presentation method with some other important unbalanced linguistic methods in the literature from the following four aspects: multi-granularity, unbalanced type, transformation and flexible scale information (See Table 5).

Table 5. The comparisons the normalized linguistic representation with other representative unbalanced linguistic representation methods.

Method	Multi-granularity	Unbalanced type	Transformation	Flexible scale information
Ours	Y	Y* (2nd)	N*	Y*
Herrera *et al.*'s[11]	Y	Y* (1nd)	Y*	N
Xu's[25] and Yu *et al.*'s[7]	Y	Y* (2nd)	Y*	N
Pedrycz and Song's[43]	N	Y* (2nd)	N	Y*
Dong *et al.*'s[27]	N	Y* (2nd)	N	Y*
Wang and Hao's[31]	N	Y* (2nd)	N	Y

Y: Yes; N: No ∗: stressed.

(1) Multi-granularity. The first four methods all deal with multi-granular linguistic terms, while the other three methods only cope with the linguistic term set with single granularity.

(2) Unbalanced type. Unbalance is considered in all mentioned methods. The first type of unbalanced linguistic term sets is considered in Herrera *et al.*'s method, while the second type is investigated in the other six methods.

(3) Transformation. In general, a multi-granular linguistic decision making problem comes with a transformation process in the literature. Since the scale domains of the linguistic term sets with different granularities have been unified in the unit

interval by the normalized scaling function, the transformation process is not needed in our method. One can do computing with word directly by using the normalized scaled semantics.

(4) Flexible scale information. Pedrycz and Song, Dong *et al.* calculate the appropriate scales of a linguistic term set by the optimization methods, which provide the selectable ways to determine the priori numerical scales for our method. Herrera *et al.* assume the multi-granular linguistic term set with special settled ordinal semantics, so do Xu and Yu *et al.* In the other four methods, the linguistic term sets have the flexible semantics that approximate to the decision maker's actual intension.

6. An Illustrative Example

Suppose that a human resource (HR) manager of a car company desires to hire a project management engineer from six candidates a_1, a_2, a_3, a_4, a_5 and a_6. Six candidates are tested in three days probation period. And at the end of each day, the HR manager uses linguistic terms to evaluate the six candidates by considering three criteria including leadership and personality (LP), computer skills (CS) and analytical and problem solving skills (APSS). The HR manager set the weight vector of the criteria as $\overrightarrow{\omega} = (0.4, 0.3, 0.3)$. And the weight vector of the three days' evaluations is given as $\overrightarrow{\lambda} = (0.2, 0.3, 0.5)$.

At the beginning of the evaluations, the HR manager expresses the semantics of a linguistic set of five terms as $\mathcal{S}_5 = \{s_0^{(5)}$: *very poor* (vp), $s_{0.6}^{(5)}$: *poor* (p), $s_{0.7}^{(5)}$: $fair$ (f), $s_{0.95}^{(5)}$: *good* (g), $s_1^{(5)}$: *very good* $(vg)\}$. On the first day, $S_3 = \{poor\ (p),\ fair\ (f),\ good\ (g)\}$ is used to express the opinions clearly; on the second day, the manager would like to deliver the views by S_5; and on the last day, as more is learned about the candidates, the manager use the term set $S_7 = \{extremely\,poor\ (ep),\ very\,poor\ (vp),\ poor\ (p),\ fair\ (f),\ good\ (g),\ very\,good\ (vg),$ $extremely\ good\ (eg)\}$ to evaluate the candidates more exactly. The linguistic evaluations delivered by the manager are shown in Table 6.

Table 6. Three days' linguistic evaluations provided by the HR manager.

	The 1st day's evaluations in S_3			The 2nd day's evaluations in S_5			The 3rd day's evaluations in S_7		
	LP	CS	$APSS$	LP	CS	$APSS$	LP	CS	$APSS$
a_1	p	f	f	vg	g	f	f	g	vg
a_2	f	g	f	g	p	g	vg	p	eg
a_3	f	f	f	p	f	g	g	eg	vp
a_4	f	f	g	f	g	vg	vg	eg	g
a_5	g	p	f	vg	f	f	f	g	p
a_6	f	f	g	g	g	p	vg	g	eg

Table 7. There days' scaled linguistic preferences.

		a_1	a_2	a_3	a_4	a_5	a_6
The	LP	$(s_{0.00}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.70}, 0\%)$
1st	CS	$(s_{0.70}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.00}, 0\%)$	$(s_{0.70}, 0\%)$
day	APSS	$(s_{0.70}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{1.00}, 0\%)$
The	LP	$(s_{1.00}, 0\%)$	$(s_{0.95}, 0\%)$	$(s_{0.60}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.95}, 0\%)$
2nd	CS	$(s_{0.95}, 0\%)$	$(s_{0.60}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.95}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.95}, 0\%)$
day	APSS	$(s_{0.70}, 0\%)$	$(s_{0.95}, 0\%)$	$(s_{0.95}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.60}, 0\%)$
The	LP	$(s_{0.70}, 0\%)$	$(s_{0.97}, 0\%)$	$(s_{0.87}, 0\%)$	$(s_{0.97}, 0\%)$	$(s_{0.70}, 0\%)$	$(s_{0.97}, 0\%)$
3rd	CS	$(s_{0.87}, 0\%)$	$(s_{0.63}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.87}, 0\%)$	$(s_{0.87}, 0\%)$
day	APSS	$(s_{0.97}, 0\%)$	$(s_{1.00}, 0\%)$	$(s_{0.40}, 0\%)$	$(s_{0.87}, 0\%)$	$(s_{0.63}, 0\%)$	$(s_{1.00}, 0\%)$
	CR	0.769	0.850	0.745	0.892	0.733	0.886

Firstly, we compute the normalized numerical scales of the linguistic sets S_3 and S_7. Using the priori scale information $\{(\frac{0}{4}, 0), (\frac{1}{4}, 0.6), (\frac{2}{4}, 0.7), (\frac{3}{4}, 0.95), (\frac{4}{4}, 1)\}$ and Eq. (6), we interpolate the key BUM function $\widehat{l}(x)$ as

$$\widehat{l}(x) = \begin{cases} 0 + 4 \times (0.6 - 0)(x - 0), & 0 \leq x < 0.25, \\ 0.6 + 4 \times (0.7 - 0.6)(x - 0.25), & 0.25 \leq x < 0.5, \\ 0.7 + 4 \times (0.95 - 0.7)(x - 0.5), & 0.5 \leq x < 0.75, \\ 0.95 + 4 \times (1 - 0.95)(x - 0.75), & 0.75 \leq x \leq 1. \end{cases}$$

Then, the other two linguistic sets are scaled as

$$\mathcal{S}_3^{\widehat{L}} = \left\{ s^{(3)}_{\widehat{l}(\frac{0}{3-1})}, s^{(3)}_{\widehat{l}(\frac{1}{3-1})}, s^{(3)}_{\widehat{l}(\frac{2}{3-1})} \right\} = \left\{ s_0^{(3)}, s_{0.7}^{(3)}, s_1^{(3)} \right\}$$

and

$$\mathcal{S}_7^{\widehat{L}} = \left\{ s^{(7)}_{\widehat{l}(\frac{i}{7-1})} \mid i = 0, 1, \ldots, 6 \right\} = \left\{ s_0^{(7)}, s_{0.4}^{(7)}, s_{0.63}^{(7)}, s_{0.7}^{(7)}, s_{0.87}^{(7)}, s_{0.97}^{(7)}, s_1^{(7)} \right\},$$

respectively. We display the scaled evaluation information in Table 7.

Secondly, we use the extended WA operator (Eq. (4)) and the weight vector $\overrightarrow{\omega} = (0.4, 0.3, 0.3)$ to aggregate the scaled evaluation information for all candidates. Taking the calculations of a_1 for example, the comprehensive evaluation of the first day is $0.4 \times (\widetilde{\Delta}_3^{\widehat{L}})^{-1}(s_{0.00}, 0\%) + 0.3 \times (\widetilde{\Delta}_3^{\widehat{L}})^{-1}(s_{0.70}, 0\%) + 0.3 \times (\widetilde{\Delta}_3^{\widehat{L}})^{-1}(s_{0.70}, 0\%) = 0.42$, the second day is $0.4 \times (\widetilde{\Delta}_5^{\widehat{L}})^{-1}(s_{1.00}, 0\%) + 0.3 \times (\widetilde{\Delta}_5^{\widehat{L}})^{-1}(s_{0.75}, 0\%) + 0.3 \times (\widetilde{\Delta}_5^{\widehat{L}})^{-1}(s_{0.70}, 0\%) = 0.895$, and the third day is $0.4 \times (\widetilde{\Delta}_7^{\widehat{L}})^{-1}(s_{0.70}, 0\%) + 0.3 \times (\widetilde{\Delta}_7^{\widehat{L}})^{-1}(s_{0.87}, 0\%) + 0.3 \times (\widetilde{\Delta}_3^{\widehat{L}})^{-1}(s_{0.97}, 0\%) = 0.832$. Then, we utilize the WA operator and the weight vector $\overrightarrow{\lambda} = (0.2, 0.3, 0.5)$ to compute the comprehensive result (CR) of the three days as $CR(a_1) = 0.2 \times 0.42 + 0.3 \times 0.895 + 0.5 \times 0.832 = 0.769$. The corresponding 2-tuple linguistic terms with different granularities are $\widetilde{\Delta}_3^{\widehat{L}}(0.769) = (s^{(3)}_{\widehat{L}(1)}, 23\%)$, $\widetilde{\Delta}_5^{\widehat{L}}(0.769) = (s^{(5)}_{\widehat{L}(2)}, 28\%)$, and $\widetilde{\Delta}_7^{\widehat{L}}(0.769) = (s^{(7)}_{\widehat{L}(3)}, 41\%)$. The other

Table 8. The results and its corresponding 2-tuple linguistic terms for each candidate.

	a_1	a_2	a_3	a_4	a_5	a_6
Value	0.769	0.850	0.745	0.892	0.733	0.886
$\overline{\mathcal{S}_3^{\widehat{L}}}$	$(s^{(3)}_{\widehat{L}(1)}, 23\%)$	$(s^{(3)}_{\widehat{L}(2)}, -50\%)$	$(s^{(3)}_{\widehat{L}(1)}, 15\%)$	$(s^{(3)}_{\widehat{L}(2)}, -36\%)$	$(s^{(3)}_{\widehat{L}(1)}, 11\%)$	$(s^{(3)}_{\widehat{L}(2)}, -38\%)$
$\overline{\mathcal{S}_5^{\widehat{L}}}$	$(s^{(5)}_{\widehat{L}(2)}, 28\%)$	$(s^{(5)}_{\widehat{L}(3)}, -40\%)$	$(s^{(5)}_{\widehat{L}(2)}, 18\%)$	$(s^{(5)}_{\widehat{L}(3)}, -24\%)$	$(s^{(5)}_{\widehat{L}(2)}, 13\%)$	$(s^{(5)}_{\widehat{L}(3)}, -26\%)$
$\overline{\mathcal{S}_7^{\widehat{L}}}$	$(s^{(7)}_{\widehat{L}(3)}, 41\%)$	$(s^{(7)}_{\widehat{L}(4)}, -13\%)$	$(s^{(7)}_{\widehat{L}(3)}, 26\%)$	$(s^{(7)}_{\widehat{L}(4)}, 22\%)$	$(s^{(7)}_{\widehat{L}(3)}, 20\%)$	$(s^{(7)}_{\widehat{L}(4)}, 16\%)$

candidates' comprehensive results can be calculated similarly. And the comprehensive results of all the candidates are shown in the last line of Table 7. The corresponding 2-tuple linguistic terms with different granularities are given in Table 8.

Finally, the total order of the six candidates is obtained as $a_4 \succ a_6 \succ a_2 \succ a_1 \succ a_3 \succ a_5$. Therefore, a_4 is the best candidate.

If the priori scale information was not considered and the traditional non-normalized balanced linguistic sets were used to calculate the comprehensive results, one should apply a transformation function to represent each linguistic evaluation value in a basic linguistic set. Our method constructs the normalized scaling function by considering the priori scale information and estimates the normalized numerical scales of the involved linguistic terms reasonably. Then the aggregation process is executed directly on the scaled linguistic set and the total order of the candidates is obtained for selecting the best one.

7. Conclusions and Further Research Topics

The unification of multi-granular linguistic information into a unique linguistic term set increases the complexity of computing with words. Meantime, it also brings out the subjectivity of determining the transformation functions. To solve these problems, this paper proposes a normalized function approach for scaling multi-granular unbalanced linguistic term sets, and also a method for constructing the normalized scale functions by the priori scale information. A new extended 2-tuple linguistic representation model is presented to participate in the computation process.

The proposed normalized scaling method has great advantages in modeling decision making problems, in which the decision maker expresses the judgments by multi-granular unbalanced linguistic terms with special semantics. The semantics of different linguistic term sets are formed into the unit interval and the transformation process is not needed. In addition, The semantics of linguistic terms in different linguistic sets are also close linked through the normalized scaling function. It is worth noting that the priori scale information in our method is articulated by decision maker directly. It is somewhat difficult for a decision maker to express the exact semantics of linguistic terms as the proper numerical values.

Several researchers have proposed the optimization approaches to solve the problem mentioned above under different frameworks. One can apply these methods for determining the priori scale information. However, the determination of numerical scales for linguistic terms needs solving a complex optimization problem. Therefore, it is necessary to explore a more convenient method to obtain the decision maker's priori scale information, which is one of our future works. In addition, the normalized scaling approach is mainly used to cope with the second type of unbalanced linguistic sets in this study. Nevertheless, the first type of unbalanced linguistic sets also frequently arises in many decision making problems. Therefore, another interesting research topic is to interpret the semantics of the first type of unbalanced linguistic sets by developing a new normalized scaling method for better solving multi-granular linguistic decision making problems.

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