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Partial ordering of information granulations: a further investigation

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Abstract: *The notion of information systems provides a convenient tool for knowledge representation of objects in terms of their attribute values, while partial ordering is usually used to research the rough monotonicity of an uncertainty measure in information systems. In this paper, we first reveal the limitations of existing partial orderings to describe information granulations in information systems with several illustrative examples. Then, a generalized partial ordering with a set-size nature is proposed to overcome their shortcoming and some of its important properties are derived. Finally, we prove that several existing information granulations all satisfy the granulation monotonicity induced by the proposed partial ordering. The presented partial ordering appears to be well suited to characterize the nature of information granulations in an information system. These results will be very helpful for studying granular computing and uncertainty in information systems.*

Keywords: information systems, partial ordering, information granulation, entropy, monotonicity

1. Introduction

The notion of information systems (sometimes called knowledge representation systems, attribute-value systems, data tables, etc.) provides a convenient tool for the representation of objects in terms of their attribute values. The rough set theory (Pawlak, 1991; Pawlak *et al.*, 1995; Pawlak & Skowron, 2007) has been introduced to deal with inexact, uncertain or vague knowledge in information systems, which has become a popular mathematical framework for pattern recognition, image processing, feature selection, neuro computing, conflict analysis, decision support, data mining and the knowledge discovery process from large data sets (Düntsch & Gediga, 1998; Guan & Bell, 1998; Gediga & Düntsch, 2001; Pawlak, 2005; Jeon *et al.*, 2006; Jensen & Shen, 2007). The use of the indiscernibility relation results in an information granulation (Yao, 2001).

According to whether or not there are missing data (null values), information systems can be classified into two categories: complete and incomplete. In a complete information system, the indiscernibility relation generated constitutes the mathematical basis of the rough set theory and induces a partition of the universe into blocks of indiscernible objects, called elementary sets (equivalence classes). For an incomplete information system, according to whether or not two objects on the universe possesses distinguished regular values under some attribute, a so-called similarity relation can be defined on the universe. However, due to the existence of null values in an incomplete system, the similarity relation, from the viewpoint of mathematics, is a tolerance relation. We know that in a knowledge representation system, the scale of the elementary knowledge units (also called knowledge granules or information granules) determines the performance of representation of general concepts. The smaller the scale of the knowledge granule, the higher the precision one can achieve.

Information granulation is an important concept of granular computing (proposed by Zadeh, 1997). The information granulation of an information system gives a measure of uncertainty about its actual structure. In general, information granulation can represent the discernibility ability of an approximate space in information systems.

Partial ordering is always used to research information granulation, measuring knowledge content, measuring the significance of an attribute and their applications in an information system. For our further investigation, as follows, we briefly review several existing partial orderings in information systems. Partial ordering has been introduced to research properties of a complete information system (Beaubouef *et al.*, 1998; Düntsch & Gediga, 1998; Klir & Wierman, 1998; Liang *et al.*, 2002; Chakik *et al.*, 2004; Liang & Li, 2005; Qian & Liang, 2008). In incomplete information systems, many researchers usually investigate some of its characters using another partial ordering (Kryszkiewiz, 1998, 1999; Liang & Xu, 2002; Leung & Li, 2003; Liang *et al.*, 2006; Qian & Liang, 2008). To consider minimal information granules in an incomplete information system, Leung and Li (2003) applied the concept of a maximal consistent block to formulate a new approximation to an object set with a higher level of accuracy. This method has been used for attribute reduction and rule acquisition in an incomplete information system. We defined a new partial ordering for this method to describe its uncertainty in incomplete information systems. However, it is worth pointing out that: these partial orderings are all essentially the second partial ordering (Qian *et al.*, 2008). In short, the partial ordering is mainly used to analyse the monotonicities of some uncertainty measures in information systems (Qian *et al.*, 2008). Nevertheless, it is unfortunate that these partial orderings have some limitations, which cannot give elaborate depictions of the relationship between knowledge in information systems. For example, if there does not exist any partial ordering from the above partial orderings between given two knowledge, then the information granulations of these knowledge cannot be characterized from the viewpoint of these existing partial orderings. To overcome these drawbacks, we need to find a new partial ordering for this target.

For a given partial ordering in a complete information system, one often considers a lattice formed by partition that is defined by an equivalence relation (Yao, 2003). In an incomplete information system, we can also consider a lattice formed by coverings, on which one can also form a partial ordering (Yao, 2001). The entropy function or the conditional entropy function will serve as a good measure (Yao, 2003).

Information granulation, as a kind of measures for the uncertainty of an information system, has been focused on widely by many researchers in recent years. Especially, several measures in complete information systems closely associated with granular computing such as information entropy, rough entropy and information granulation and their relationships are discussed by Liang and Shi (2004). Recently, Liang *et al.* (2006) extended these measures to an incomplete information system. Combination granulation and combination entropy in information systems are proposed to calculate the uncertainty of an information system (Qian & Liang, 2008), in which the gain function possesses an intuitionistic knowledge content nature. It should be mentioned that these above measures all satisfy the rough monotonicity induced by the second partial ordering.

In this paper, we focus on the limitations of the above partial orderings for measuring the uncertainty of an information system and propose a new partial ordering that can overcome these limitations. Based on the proposed partial ordering, we will research and give quantitative measures that characterize the order partially.

The paper is organized as follows: in Section 2, we review some basic concepts, such as complete information systems, incomplete information systems and the maximal consistent block technique. By analysing structures of existing partial orderings, in Section 3, we analyse the limitations of these partial orderings to describe information granulation in information systems. In Section 4, we define a novel partial ordering, and show how it overcomes the limitation of existing partial orderings by several illustrative examples. In Section 5, we prove that several existing information granulations all satisfy granulation monotonicity determined by the proposed partial ordering. Finally, Section 6 concludes the whole paper.

2. Information systems

An information system is a pair $S=(U,A)$, where

1. U is a non-empty finite set of objects;
2. A is a non-empty finite set of attributes; and
3. for every $a \in A$, there is a mapping a, a: $U \rightarrow V_a$, where U_a is called the value set of a.

For an information system $S=(U,A)$, if $\forall a \in A$, every element in V_a is a definite value, then S is called a complete information system.

Each subset of attributes $P \subseteq A$ determines a binary indistinguishable relation $IND(P)$ as follows:

$$IND(P)=\{(u,v) \in U \times U | \forall a \in P, a(u)=a(v)\}$$

It is easily shown that

$$IND(P)= \cap_{a \in P} IND(\{a\})$$

$U/IND(P)$ constitutes a partition of U. $U/IND(P)$ is called a knowledge in U; every equivalence class is called a knowledge granule or information granule (Yao, 2000). Information granulation, in a sense, denotes the average measure of information granules (equivalence classes) in P. In general, we denote the knowledge induced by $P \subseteq A$ by U/P.

Table 1: *A complete information system about cars*

Car	*Price*	*Mileage*	*Size*	*Max-Speed*
u_1	High	Low	Full	Low
u_2	Low	High	Full	Low
u_3	Low	Low	Compact	Low
u_4	High	High	Full	High
u_5	High	High	Full	High
u_6	Low	High	Full	Low

Example 2.1

Consider the descriptions of several cars in Table 1.

This is a complete information system, where $U=\{u_1,u_2,u_3,u_4,u_5\}$, and $A=\{a_1,a_2,a_3,a_4\}$ with a_1 being the Price, a_2 the Mileage, a_3 the Size and a_4 the Maximum speed. By computing, it follows that

$$U/IND(A)=\{\{u_1\},\{u_2,u_6\},\{u_3\},\{u_4,u_5\}\}$$

It may happen that some of the attribute values for an object are missing. For example, in medical information systems, there may be a group of patients for whom it is impossible to perform all the required tests. These missing values can be represented by the set of all possible values for the attribute or equivalence by the domain of the attribute. To indicate such a situation, a distinguished value, a so-called null value, is usually assigned to those attributes.

If V_a contains a null value for at least one attribute $a \in A$, then S is called an incomplete information system (Kryszkiewiz, 1998, 1999; Qian & Liang, 2008); otherwise, it is complete. Further on, we will denote the null value by $*$.

Let $S=(U,A)$ be an information system, $P \subseteq A$ an attribute set. We define a binary relation on U as follows:

$$SIM(P) = \{(u, v) \in U \times U | \forall a \in P,$$

$$a(u) = a(v) \; or \; a(u) = * \; or \; a(v) = *\}$$

In fact, $SIM(P)$ is a tolerance relation on U; the concept of a tolerance relation has a wide variety of applications in classification (Slowinski & Vanderpooten, 2000; Chin *et al.*, 2003; Liang & Qian, 2006).

It can be easily shown that

$$SIM(P) = \cap_{a \in P} SIM(\{a\})$$

Let $U/SIM(P)$denote the family sets $\{S_P(u)|u \in U\}$ as the classification induced by P, where $S_P(u) = \{v \in U|(u, v) \in SIM(P)\}$ is the tolerance class determined by the object u. It should be noticed that the tolerance classes in $U/SIM(P)$ do not constitute a partition of U in general. They constitute a cover of U, i.e., $S_P(u) \neq Ø$ for every $u \in U$ and $\cup_{u \in U} S_P(u) = U$.

Of course, $SIM(P)$ degenerates into an equivalence relation in a complete information system.

Example 2.2

Consider the descriptions of several cars in Table 2. This is an incomplete information system, where $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, and $A = \{a_1, a_2, a_3, a_4\}$ with a_1 – Price, a_2 – Mileage, a_3 – Size, a_4 – Max-Speed. By computing, it follows that

$$U/SIM(A) = \{S_A(u_1), S_A(u_2), S_A(u_3), S_A(u_4),$$

$$S_A(u_5), S_A(u_6)\}$$

where $S_A(u_1) = \{u_1\}$, $S_A(u_2) = \{u_2, u_6\}$, $S_A(u_3) = \{u_3\}$, $S_A(u_4) = \{u_4, u_5\}$, $S_A(u_5) = \{u_4, u_5, u_6\}$, $S_A(u_6) = \{u_2, u_5, u_6\}$.

However, tolerance classes are not the minimal units for describing knowledge or information in incomplete information systems (Leung & Li, 2003).

Let $S = (U, A)$ be an information system, $P \subseteq A$ an attribute set and $X \subseteq U$ a subset of objects. We say X is consistent with respect to P if $(u, v) \in SIM(P)$ for any u, $v \in X$. If there does not exist a subset $Y \subseteq U$ such that $X \subset Y$, and Y is consistent with respect to P, then X is called a maximal consistent block of P. Obviously, in a maximal consistent block, all objects are not indiscernible with available information provided by a similarity relation (Leung & Li, 2003).

Table 2: *An incomplete information system about cars*

Car	*Price*	*Mileage*	*Size*	*Max-Speed*
u_1	High	Low	Full	Low
u_2	*	*	Full	Low
u_3	Low	*	Compact	Low
u_4	High	*	Full	High
u_5	*	*	Full	High
u_6	Low	High	Full	*

Henceforth, we denote the set of all maximal consistent blocks determined by $P \subseteq A$ as C_P, and the set of all maximal consistent blocks of P, which includes some object $u \in U$, is denoted as $C_P(u)$.

It is obvious that $X \in C_P$ if and only if $X = \cap_{u \in X} S_P(u)$ (Leung & Li, 2003).

Example 2.3

Computing all maximal consistent blocks of A in Table 2. By computing, from Example 2.2, we have that

$$C_A = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\}$$

where C_A is the set of all maximal consistent blocks determined by A on U.

3. The limitation of partial ordering $\preceq$

In a complete information system, partial ordering $\preceq_1$ is usually used to analyse the inclusion relationship on all partitions on the universe (Beaubouef *et al.*, 1998; Klir & Wierman, 1998; Liang *et al.*, 2002; Chakik *et al.*, 2004), while partial ordering $\preceq$ is often used to characterize the inclusion relationship on all covers in incomplete information systems (Kryszkiewiz, 1998, 1999; Liang *et al.*, 2006; Qian & Liang, 2008). For maximal consistent blocks in incomplete information systems, the above two partial orderings cannot be directly used to distinguish two attribute subsets in view of granular computing. For this reason, we will define partial ordering $\preceq_2$ to the maximal consistent block technique.

In this section, by analysing the structure of partial orderings $\preceq_1$, $\preceq_2$ and $\preceq$, we reveal the limitation of partial ordering $\preceq$ to describe information granulation in information systems by several illustrative examples.

Definition 3.1 (Yao, 2001; Liang *et al.*, 2002) Let $S=(U,A)$ be a complete information system, $P,Q\subseteq A$, $U/IND(P)=\{P_1, P_2, \ldots, P_m\}$ and $U/IND(Q)=\{Q_1, Q_2, \ldots, Q_n\}$. We define partial ordering $\preceq_1$ as follows:

$P\preceq_1 Q \Longleftrightarrow$ For every $P_i\in U/IND(P)$, there exists $Q_j\in U/IND(Q)$ such that $P_i\subseteq Q_j$.

If $P\preceq_1 Q$ and $P\neq Q$, i.e., for some $P_{i_0}\in U/IND(P)$, there exists $Q_{j_0}\in U/IND(Q)$ such that $P_{i_0}\subset Q_{j_0}$, we denote it as $P\prec_1 Q$.

If $P\preceq_1 Q$, we say that Q is coarser than P (or P is finer than Q). If $P\prec_1 Q$, we say that Q is strictly coarser than P (or P is strictly finer than Q).

Example 3.1

Analyse the relationship between two attribute subsets in Table 1. Let $P=\{Price,\ Mileage\}$, $Q=\{Price\}$ be two attribute subsets in Table 1. By computing, we have that

$$U/IND(P)=\{\{u_1\},\{u_2,u_6\},\{u_3\},\{u_4,u_5\}\}$$

$$U/IND(Q)=\{\{u_1,u_4,u_5\},\{u_2,u_3,u_6\}\}$$

It is obvious that $P\prec_1 Q$.

Example 3.1 shows that the partition $IND(Q)$ is much coarser than the partition induced by P, in which each equivalence class is the union of some equivalence classes in $U/IND(P)$. For instance, the set $\{u_1,u_4,u_5\}=\{u_1\}\cup\{u_4,u_5\}$. In other words, the partial ordering $\prec_1$ is an approach to characterizing information granulations in complete information systems. However, the partial ordering $\preceq_1$ cannot appropriately describe information granulation in complete information systems. Its limitation can be revealed by the following example.

Example 3.2

Consider two partitions as follows:

$$U/IND(P)=\{\{u_1\},\{u_2,u_6\},\{u_3\},\{u_4,u_5\}\}$$
$$U/IND(Q)=\{\{u_1,u_3,u_5\},\{u_2,u_4,u_6\}\}$$

From Definition 3.1, it is obvious that $P\not\preceq_1 Q$, whereas one can intuitively know that $U/IND(Q)$ is much coarser than $U/IND(P)$ and the information granulation induced by Q should be bigger than that induced by P. That is to say, this situation cannot be characterized by the partial ordering $\prec_1$. Therefore, a new partial ordering is needed for depicting this case.

From Definition 3.1, it can be easily seen that Definition 3.1 is not applicable to an incomplete information system because the tolerance classes cannot be induced by an equivalence relation on the universe. Hence, the partial ordering $\preceq$ was introduced to incomplete information systems for depicting the relationship between two attribute sets.

Definition 3.2 (Liang *et al.*, 2006) Let $S=(U,A)$ be an incomplete information system, $P,Q\subseteq A$, $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$, $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$. We define partial ordering $\preceq$ as follows:

$$P\preceq Q\Leftrightarrow S_P(u_i)\subseteq S_Q(u_i),\ \forall i\in\{1,2,\ldots,|U|\}$$

If $P\preceq Q$, we say that Q is coarser than P (or P is finer than Q). If $P\preceq Q$ and $P\neq Q$, we say that Q is strictly coarser than P (or P is strictly finer than Q) and denoted by $P\prec Q$. In fact, $P\prec Q\Leftrightarrow$ for $\forall i\in\{1,2,\ldots,|U|\}$, we have that $S_P(u_i)\subseteq S_Q(u_i)$, and there exists $j\in\{1,2,\ldots,|U|\}$, such that $S_P(u_j)\subset S_Q(u_j)$.

Example 3.3

Analyse the relationship between two attribute subsets in Table 2. Let $P=\{Price,\ Mileage\}$, $Q=\{Price\}$ be two attribute subsets in

Table 2. By computing, we have that

$$U/SIM(P) = \{\{u_1, u_4, u_5\}, \{u_2, u_5, u_6\}, \{u_3\}, \{u_1, u_4, u_5\}, \{u_1, u_2, u_4, u_5, u_6\}, \{u_2, u_5, u_6\}\}$$
$$U/SIM(Q) = \{\{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_5, u_6\} \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_2, u_3, u_5, u_6\}\}$$

It is obvious that $P \prec Q$from Definition 3.2.

Example 3.3 indicates that the cover $U/SIM(Q)$ is much coarser than the cover induced by P, in which each tolerance class is included in the corresponding tolerance class in $U/SIM(Q)$, i.e., $S_P(u) \subseteq S_Q(u)$ for any $u \in U$. That is to say, the partial ordering $\preceq$ is an approach to characterizing information granulations in incomplete information systems. However, the partial ordering $\preceq$ also cannot describe appropriately information granulation in incomplete information systems. It can be understood by the following example.

Example 3.4

Consider the following two covers.

$$U/SIM(P) = \{\{u_1, u_4, u_5\}, \{u_2, u_5, u_6\}, \{u_3\}, \{u_1, u_4, u_5\}, \{u_1, u_2, u_4, u_5, u_6\}, \{u_2, u_5, u_6\}\}$$
$$U/SIM(Q) = \{\{u_1, u_2, u_4, u_5\}, \{u_2, u_3, u_5, u_6\} \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_2, u_3, u_5, u_6\}\}$$

Since the tolerance class $\{u_1, u_2, u_4, u_5\}$ in $U/SIM(Q)$ cannot be obtained by combining some tolerance classes in $U/SIM(P)$, one has that $P \not\preceq Q$. By comparing the sizes of their tolerance classes, however, one can know that $U/SIM(Q)$ is much coarser than $U/SIM(P)$, in which the cardinality of the tolerance class of each object in $U/SIM(P)$ is much bigger than that of the tolerance class of that object in the universe. That is to say, this situation cannot be characterized by partial ordering $\prec$. In order to essentially characterize information granulation in incomplete information systems, one needs to find a new partial ordering for this target.

In fact, the partial ordering $\preceq$ can degenerate into partial ordering $\preceq_1$ in a complete information system.

Theorem 3.1 Partial ordering $\preceq_1$ is a special instance of partial ordering $\preceq$.

Proof It is straightforward. ■

Similar to the definitions of $\preceq_1$ and $\preceq$, we define another partial ordering in order to discuss the properties of the maximal consistent block technique in incomplete information systems.

Definition 3.3 Let $S = (U, A)$ be an incomplete information system, $P, Q \subseteq A$, $C_P = \{P^1, P^2, \ldots, P^m\}$, $C_Q = \{Q^1, Q^2, \ldots, Q^n\}$. We define partial ordering $\preceq_2$ as follows: $P \preceq_2 Q \Leftrightarrow$ for every $P^i \in C_P$, there exists $Q^j \in C_Q$ such that $P^i \subseteq Q^j$.

If $P \preceq_2 Q$ and $P \neq Q$, i.e., for some $P^{i_0} \in C_P$, there exists $Q^{j_0} \in C_Q$ such that $P^{i_0} \subset Q^{j_0}$, we denote it by $P \prec_2 Q$.

If $P \preceq_2 Q$, we say that Q is coarser than P (or P is finer than Q). If $P \prec_2 Q$, we say that Q is strictly coarser than P (or P is strictly finer than Q).

From the definition of the above partial ordering, one can see that this definition is a natural generalization of the partial ordering $\preceq$ in incomplete systems. It is illuminated by the following example.

Example 3.5

Analyse the relationship between two attribute subsets in Table 2. Let $P = A$, $Q = \{Price, Size, Max\text{-}Speed\}$ be two attribute subsets in Table 2. By computing, we have that

$$C_P = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\}$$
$$C_Q = \{\{u_1, u_2, u_6\}, \{u_3\}, \{u_4, u_5, u_6\}\}$$

From the definition of $\preceq_2$, it is follows that $P \prec_2 Q$.

Example 3.5 shows that C_Q is much coarser than C_P, in which each maximal consistent block is the union of some maximal consistent blocks in C_P. In other words, for every $P^i \in C_P$, there exists $Q^j \in C_Q$ such that $P^i \subseteq Q^j$. However, partial ordering $\preceq_2$ also cannot describe appropriately information granulation in incomplete information systems. Its limitation can be revealed by the following example.

Example 3.6
Consider two maximal consistent blocks as follows:

$$C_P = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\}$$
$$C_Q = \{\{u_1, u_3, u_6\}, \{u_2\}, \{u_4, u_5, u_6\}\}$$

From Definition 3.3, we have that $P \not\preceq_2 Q$ because $\{u_1, u_3, u_6\}$ in C_Q cannot be constructed by several maximal consistent blocks in C_P. By comparing the sizes of these maximal consistent blocks, however, one can see that C_P is much finer than C_Q in fact. In other words, this situation cannot be well characterized by partial ordering $\preceq_2$. In this situation, we need to consider their inner structures of information granules in a given incomplete information system and define a more reasonable partial ordering for characterizing the relationship among information granulations.

In fact, the partial ordering $\preceq_2$ can be transformed to the partial ordering $\preceq$ in incomplete information systems. It can be illustrated by the following theorem.

Theorem 3.2 Partial ordering $\preceq_2$ is a special instance of partial ordering $\preceq$.

Proof Let $S = (U, A)$ be an incomplete information system, $P, Q \subseteq A$ with $P \preceq_2 Q$, $C_P = \{P^1, P^2, \ldots, P^m\}$ and $C_Q = \{Q^1, Q^2, \ldots, Q^n\}$. It follows from the definition of $\preceq_2$ that for arbitrary $P^i \in C_P$, there exists $Q^i \in C_Q$ such that $P^i \subseteq Q^j$.

Next, we prove that $S_P(u) \subseteq S_Q(u)$ for $\forall u \in U$. Assume that $C_P(u) = \{X_1, X_2, \ldots, X_m\}$ and $C_Q(u) = \{Y_1, Y_2, \ldots, Y_n\}$. In addition, we know that $S_P(u) = \cup\{X_k \in C_P | X_k \subseteq S_P(u)\} = \cup\{X_k \in C_P(u)\} (k \leq m)$ and $S_Q(u) = \cup\{Y_t \in C_Q | Y_t \subseteq S_Q(u)\} = \cup\{Y_t \in C_Q(u)\} (t \leq n)$ from property 4 in the literature (Leung & Li, 2003). From the definition of a maximal consistent block, we have that $u \in C_P(u)$, $u \in C_Q(u)$, $u \notin C_P - C_P(u)$ and $u \notin C_Q - C_Q(u)$. Hence, it follows from $P \preceq_2 Q$ that for arbitrary $X_k \in C_P(u)$, there exist $Y_t \in C_Q(u)$ such that $X_k \subseteq Y_t$. Thus, for arbitrary $u \in U$, we have that

$$\begin{aligned} S_P(u) &= \cup\{X_k \in C_P | X_k \subseteq S_P(u)\} \\ &= \bigcup_{k=1}^{m} X_k \subseteq \bigcup_{t=1}^{n} Y_t \\ &= \cup\{Y_t \in C_Q | Y_t \subseteq S_Q(u)\} \\ &= S_Q(u) \end{aligned}$$

that is $P \preceq Q$.

Hence, partial ordering $\preceq_2$ is a special instance of partial ordering $\preceq$. This completes the proof. ■

From Theorems 3.1 and 3.2, we know that partial orderings $\preceq_1$ and $\preceq_2$ are all induced to the partial ordering $\preceq$. So far, this partial ordering has been widely used to describe the uncertainty of information systems. However, from the above analyses, this partial ordering cannot well characterize the essence of uncertainty in information systems.

4. A generalized partial ordering $\trianglerighteq$

In this section, we will introduce a new partial ordering $\trianglerighteq$ with a set-size character to overcome the limitation of partial ordering $\preceq$.

For convenience, in the sequel, we denote the covering $U/SIM(A)$ by $K(A)$.

Definition 4.1 Let $S = (U, A)$ be an information system, $P, Q \subseteq A$, $K(P) = \{S_P(u) | u \in U\}$ and $K(Q) = \{S_Q(u) | u \in U\}$. We define a binary relation $\trianglerighteq$ with a set size character as follows:

$$P \trianglerighteq Q \Leftrightarrow \text{for} K(P) = \{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$$

there exists a sequence $K'(Q)$of $K(Q)$, where $K'(Q) = \{S_Q(u'_1), S_Q(u'_2), \ldots, S_Q(u'_U)\}$, such that $|S_P(u_i)| \leq |S_Q(u'_i)|$.

If there exists a sequence $K'(Q)$ of $K(Q)$ such that $|S_P(u_i)| \leq |S_Q(u'_i)|$ and $|S_P(u_{i_0})| < |S_Q(u_{i_0})|$ for some $u_{i_0} \in U$, then we will state that P is strict granulation finer than Q, and denote it by $P \triangleright Q$; if there exists a sequence $K'(Q)$ of $K(Q)$ such that $|S_P(u_i)| = |S_Q(u'_i)|$ for arbitrary $u \in U$, then we will state that P is granulation equal to Q, and denote it by $P \approx Q$.

Similar to existing partial orderings in complete/incomplete information systems, the proposed partial ordering also forms a lattice, which can degenerated into each of existing partial orderings with various restrictions.

Let $S = (U, A)$ be an information system, and $\tilde{K} = \{K(P) | P \subseteq A\}$ be all knowledge induced by A on the universe U. One can obtain the following theorem.

Theorem 4.1 $(\tilde{K}, \trianglerighteq)$ is a partial set.

Proof Let $P, Q, R \subseteq A$, $K(P) = \{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$, $K(Q) = \{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$, and $K(R) = \{S_R(u_1), S_R(u_2), \ldots, S_R(u_{|U|})\}$.

1. For arbitrary $u \in U$, it is obvious $|S_P(u)| = |S_P(u)|$; hence, $P \trianglerighteq P$.
2. Suppose $P \trianglerighteq Q$ and $Q \trianglerighteq P$. From the above definition, we obtain that

$P \trianglerighteq Q \Leftrightarrow$ for $K(P)$, there exists a sequence $K'(Q)$ of $K(Q)$ such that $|S_P(u_i)| \leq |S_Q(u'_i)|$, where $K'(Q) = \{S_Q(u'_1), S_Q(u'_2), \ldots, S_Q(u'_U)\}$; $Q \trianglerighteq P \Leftrightarrow$ for $K(Q)$, there exists a sequence $K'(P)$ of $K(P)$ such that $|S_Q(u_i)| \leq |S_P(u'_i)|$, where $K'(P) = \{S_P(u'_1), S_P(u'_2), \ldots, S_P(u'_U)\}$.

Therefore, we have that

$$\sum_{i=1}^{|U|} |S_P(u_i)| \leq \sum_{i=1}^{|U|} |S_Q(u'_i)| = \sum_{i=1}^{|U|} |S_Q(u_i)| \leq \sum_{i=1}^{|U|} |S_P(u'_i)|$$

And, since $\sum_{i=1}^{|U|} |S_P(u_i)| = \sum_{i=1}^{|U|} |S_P(u'_i)|$, one can obtain that $\sum_{i=1}^{|U|} |S_P(u_i)| = \sum_{i=1}^{|U|} |S_Q(u'_i)|$. Considering $|S_P(u_i)| \leq |S_Q(u'_i)|$, hence $|S_P(u_i)| = |S_Q(u'_i)| (i \leq |U|)$. Thus, $P \approx Q$ holds.

3. Suppose $P \trianglerighteq Q$ and $Q \trianglerighteq R$. From the above definition, we obtain that

$P \trianglerighteq Q \Leftrightarrow$ for $K(P)$, there exists a sequence $K'(Q)$ of $K(Q)$ such that $|S_P(u_i)| \leq |S_Q(u'_i)|$, where $K'(Q) = \{S_Q(u'_1), S_Q(u'_2), \ldots, S_Q(u'_U)\}$; $Q \trianglerighteq R \Leftrightarrow$ for $K(Q)$, there exists a sequence $K'(R)$ of $K(R)$ such that $|S_Q(u_i)| \leq |S_R(u'_i)|$, where $K'(R) = \{S_R(u'_1), S_R(u'_2), \ldots, S_R(u'_U)\}$.

Hence, for the sequence $K'(Q)$, there exists a sequence $K''(R)$ of $K(R)$ such that $|S_Q(u'_i)| \leq |S_R(u''_i)|$, where $K''(R) = \{S_R(u''_1), S_R(u''_2), \ldots, S_R(u''_U)\}$.

Therefore, for $K(P)$, there exists a sequence $K''(R)$ of $K(R)$ such that $|S_P(u_i)| \leq |S_R(u''_i)|$, i.e., $P \trianglerighteq R$.

Thus, $(\tilde{K}, \trianglerighteq)$ is a partial set. This completes the proof. ■

From the above theorem, it follows that the binary relation $\trianglerighteq$ on the power set of A is a partial ordering. The following example shows the mechanism of partial ordering $\trianglerighteq$.

Example 4.1

Analyse the relationship between two attribute subsets in Table 2. Let $P = \{Max\text{-}Speed\}$, $Q = \{Price\}$ be two attribute subsets in Table 2. By computing, we have that

$$K(P) = \{\{u_1, u_2, u_3, u_6\}, \{u_1, u_2, u_3, u_6\}, \{u_1, u_2, u_3, u_6\}\{u_4, u_5, u_6\}, \{u_4, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}\}$$

$$K(Q) = \{\{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_2, u_3, u_5, u_6\}\}$$

Assume that $K'(Q) = \{\{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}\}$.

Obviously, there exists a sequence $K'(Q)$ of $K(Q)$ such that $|S_P(u_i)| \leq |S_Q(u'_i)| (S_P(u_i) \in K(P), S_Q(u'_i) \in K'(Q))$. Since $|S_P(u_3)| = 4 < 6 = |S_Q(u'_3)|$, we have that $P \triangleright Q$.

In the sequel, how the partial ordering $\trianglerighteq$ overcomes the limitation of partial ordering $\preceq$ will be illustrated by the following examples.

Example 4.2 (Continued from Example 3.2)

$$U/IND(P) = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}\}$$
$$U/IND(Q) = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$$

In incomplete information systems, they can be expressed as

$$K(P) = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_4, u_5\}, \{u_2, u_6\}\}$$
$$K(Q) = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$$

Suppose that

$$K'(Q) = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$$

Clearly, for arbitrary $i \in U$, we have that $|S_P(u_i)| < |S_Q(u'_i)|$. Hence, it follows from Definition 4.1 that $P \triangleright Q$ holds.

Therefore, this situation can be characterized by the partial ordering $\trianglerighteq$. However, in Example 3.2, one has that $P \npreceq_1 Q$ for the attribute sets P and Q. That is to say, the partial ordering $\trianglerighteq$ can characterize the situation that cannot be depicted by the partial ordering $\preceq_1$. Hence, the partial ordering $\trianglerighteq$ is much better than the partial ordering $\preceq_1$ for characterizing the relationship among knowledge from a complete information system.

Clearly, one can obtain the following theorem.

Theorem 4.2 Partial ordering $\preceq_1$ can be induced to partial ordering $\trianglerighteq$.

Example 4.3 (continued from Example 3.4)

$$U/SIM(P) = \{\{u_1, u_4, u_5\}, \{u_2, u_5, u_6\}, \{u_3\}, \{u_1, u_4, u_5\}, \{u_1, u_2, u_4, u_5, u_6\}, \{u_2, u_5, u_6\}\}$$
$$U/SIM(Q) = \{\{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_2, u_3, u_5, u_6\}\}$$

In fact, $K(P) = U/SIM(P)$and $K(Q) = U/SIM(Q)$ in incomplete information systems.

Let $K'(Q) = K(Q) = U/SIM(Q)$.

Clearly, for arbitrary $i \in U$, we have that $|S_P(u_i)| < |S_Q(u'_i)|$. Hence, it follows from Definition 4.1 that $P \triangleright Q$ holds.

Therefore, this situation can also be characterized by the partial ordering $\trianglerighteq$. But, in Example 3.4, we know that $P \npreceq Q$ for the attribute sets P and Q. In other words, the partial ordering $\trianglerighteq$ can characterize the situation that cannot be depicted by the partial ordering $\preceq$. Therefore, the partial ordering $\trianglerighteq$ is much better than the partial ordering $\preceq$ for depicting the relationship between knowledge from an incomplete information system.

From the above example, it is easy to obtain the following theorem.

Theorem 4.3 Partial ordering $\preceq$ can be induced to partial ordering $\trianglerighteq$.

Example 4.4 (continued from Example 3.6)

$$C_P = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_5, u_6\}\},$$
$$C_Q = \{\{u_1, u_3, u_6\}, \{u_2\}, \{u_4, u_5, u_6\}\}$$

Since the same maximal consistent blocks can induce multi tolerance relations in incomplete information systems, we only employ the following instance to describe the partial ordering $\trianglerighteq$

$$K(P) = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_4, u_5, u_6\} \{u_2, u_5, u_6\}\}$$
$$K(Q) = \{\{u_1, u_3, u_6\}, \{u_2\}, \{u_1, u_3, u_6\}, \{u_4, u_5, u_6\}, \{u_4, u_5, u_6\}, \{u_1, u_2, u_3, u_4, u_5, u_6\}\}$$

Suppose that

$$K'(Q) = \{\{u_2\}, \{u_1, u_3, u_6\}, \{u_1, u_3, u_6\}, \{u_4, u_5, u_6\}, \{u_4, u_5, u_6\}, \{u_1, u_3, u_4, u_5, u_6\}\}$$

By computing, we have that

$$|\{u_1\}| = 1 = |\{u_2\}|$$
$$|\{u_2, u_6\}| = 2 < 3 = |\{u_1, u_3, u_6\}|$$
$$|\{u_3\}| = 1 < 3 = |\{u_1, u_3, u_6\}|$$
$$|\{u_4, u_5\}| = 2 < 3 = |\{u_4, u_5, u_6\}|$$
$$|\{u_4, u_5, u_6\}| = 3 = |\{u_4, u_5, u_6\}|$$
$$|\{u_2, u_5, u_6\}| = 3 < 5 = |\{u_1, u_3, u_4, u_5, u_6\}|$$

Hence, from Definition 4.1, it follows that $P \triangleright Q$. In Example 3.6, nevertheless, we have that $P \not\preceq_2 Q$ for the attribute sets P and Q. In other words, the partial ordering $\trianglerighteq$ can characterize the situation that cannot be described by the partial ordering $\preceq_2$. Therefore, the partial ordering $\trianglerighteq$ is much better than the partial ordering $\preceq_2$ for depicting the relationship between knowledge in the context of maximal consistent blocks.

In fact, this mechanism can be illustrated by the following theorem.

Theorem 4.4 Partial ordering $\preceq_2$ can be induced to partial ordering $\trianglerighteq$.

Proof Let $S = (U, A)$ be an incomplete information system, $P, Q \subseteq A$ with $P \preceq_2 Q$, $C_P = \{P^1, P^2, \ldots, P^m\}$ and $C_Q = \{Q^1, Q^2, \ldots, Q^n\}$. It follows from the definition of $\preceq_2$ that for arbitrary $P^i \in C_P$, there exists $Q^i \in C_Q$ such that $P^i \subseteq Q^j$ and $|P^i| \leq |Q^j|$.

Next, we prove that $|S_P(u)| \leq |S_Q(u)|$, $\forall u \in U$. Assume that $C_P(u) = \{X_1, X_2, \ldots, X_m\}$ and $C_Q(u) = \{Y_1, Y_2, \ldots, Y_n\}$.

We know that $S_P(u) = \cup\{X_k \in C_P | X_k \subseteq S_P(u)\} = \cup\{X_k \in C_P(u)\}$ $(k \leq m)$ and $S_Q(u) = \cup\{Y_t \in C_Q | Y_t \subseteq S_Q(u)\} = \cup\{Y_t \in C_Q(u)\}$ $(t \leq n)$ from property 4 in the literature (Leung & Li, 2003). From the definition of the maximal consistent block, we have that $u \in C_P(u)$, $u \in C_Q(u)$, $u \notin C_P - C_P(u)$ and $u \notin C_Q - C_Q(u)$. Hence, it follows from $P \preceq_2 Q$ that for arbitrary $X_k \in C_P(u)$, there exists $Y_t \in C_Q(u)$ such that $X_k \subseteq Y_t$. Therefore,

$$\bigcup_{k=1}^{m} X_k \subseteq \bigcup_{t=1}^{n} Y_t \text{ and } \left|\bigcup_{k=1}^{m} X_k\right| \leq \left|\bigcup_{t=1}^{n} Y_t\right|$$

Thus, we have that

$$\begin{aligned} |S_P(u)| &= \left|\bigcup\{X_i \in C_P(u)\}\right| \\ &= \left|\bigcup_{k=1}^{m} X_k\right| \leq \left|\bigcup_{t=1}^{n} Y_t\right| \\ &= \left|\bigcup\{Y_y \in C_Q(u)\}\right| = |S_Q(u)| \end{aligned}$$

Hence, $|S_P(u)| \leq |S_Q(u)|$, $\forall u \in U$, i.e., $P \trianglerighteq Q$ holds. This completes the proof. ■

Theorem 4.2 shows that partial ordering $\preceq_2$ is also a special instance of partial ordering $\trianglerighteq$.

In what follows, we employ an example to exemplify the merit of the proposed partial ordering $\trianglerighteq$.

Example 4.5

Consider a given information system (see Table 3).

This is an information system, where $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $A = \{a_1, a_2, a_3\}$. By computing, it follows that

$$U/IND(\{a_1\}) = \{\{u_1\}, \{u_2, u_5\}, \{u_3\}, \{u_4\}, \{u_6\}\}$$
$$U/IND(\{a_2\}) = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}\}$$
$$U/IND(\{a_3\}) = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$$

From this computation, one cannot obtain the relationships between the above three partitions in terms of the existing partial orderings. In fact, we have that $\{a_1\} \not\preceq_1 \{a_2\} \not\preceq_1 \{a_3\}$, $\{a_1\} \not\preceq \{a_2\} \not\preceq \{a_3\}$ and $\{a_1\} \not\preceq_2 \{a_2\} \not\preceq_2 \{a_3\}$. In other words, these existing three partial orderings cannot characterize the relationship between the three partitions.

In fact, the above partitions can also be denoted by the following knowledge:

$$K(\{a_1\}) = \{\{u_1\}, \{u_2, u_5\}, \{u_3\}, \{u_4\}, \{u_2, u_5\}, \{u_6\}\}$$
$$K(\{a_2\}) = \{\{u_1\}, \{u_2, u_6\}, \{u_3\}, \{u_4, u_5\}, \{u_4, u_5\}, \{u_2, u_6\}\}$$
$$K(\{a_3\}) = \{\{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}, \{u_1, u_3, u_5\}, \{u_2, u_4, u_6\}\}$$

Table 3: *An information system with three attributes*

Objects	*a₁*	*a₂*	*a₃*
u_1	0	0	0
u_2	1	1	1
u_3	2	2	0
u_4	3	3	1
u_5	1	3	0
u_6	4	1	1

If we adopt the proposed partial ordering, it is clear that

$$\{a_1\} \trianglerighteq \{a_2\} \trianglerighteq \{a_3\}$$

That is to say, the proposed partial ordering may be much better than existing partial orderings for characterizing the relationship between knowledge in a complete/incomplete information system.

From the above discussions and analyses, one can draw a conclusion that the partial ordering $\trianglerighteq$ may be a much better description of characterizing the relationship between knowledge in a complete/incomplete information system. As a result, a uniform representation of the partial orderings is obtained for characterizing the relationship among knowledge information systems, which may lead to a strategy for dealing with information granulations in some real-word applications that involve complete information, incomplete information and maximal consistent blocks all together.

5. Granulation monotonicities of some existing information granulations

Conveniently, partial ordering $\trianglerighteq$ can be called a granulation partial ordering; the monotonicity of information granulation induced by it can be called granulation monotonicity. In recent years, some different kinds of information granulations were defined, and their rough monotonicities had been systemically researched as well (Liang & Shi, 2004; Liang & Qian, 2006; Liang *et al*., 2006; Qian & Liang, 2008). In this section, we will investigate granulation monotonicities of these measures.

Definition 5.1 (Liang & Shi, 2004) Let $S = (U, A)$ be a complete information system, $U/IND(A) = \{P_1, P_2, \ldots, P_m\}$. An information granulation of A is defined by

$$GK(A) = \frac{1}{|U|^2}\sum_{i=1}^{m} |P_i|^2$$

where $\frac{1}{|U|} \leq GK(A) \leq 1$and $\sum_{i=1}^{m} |P_i|^2$ is the cardinality of the equivalence relation $\bigcup_{i=1}^{m} (P_i \times P_i)$ determined by A.

Theorem 5.1 Let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. If $P \triangleright Q$, then $GK(P) < GK(Q)$.

Proof Let $P, Q \subseteq A$. In complete information systems, $U/IND(P) = \{P_1, P_2, \ldots, P_m\}$, $P_i = \{u_{i1}, u_{i2}, \ldots, u_{is_i}\}$, where $|P_i| = s_i$ and $\sum_{i=1}^{m} s_i = |U|$; $U/IND(Q) = \{Q_1, Q_2, \ldots, Q_n\}$, $Q_j = \{u_{j1}, u_{j2}, \ldots, u_{jt_j}\}$, where $|Q_j| = t_j$ and $\sum_{j=1}^{n} t_j = |U|$. Suppose that $K(P) = \{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$ and $K(Q) = \{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$ in incomplete information systems. Therefore, we have that

$$P_i = S_P(u_{i1}) = S_P(u_{i2}) = \cdots = S_P(u_{is_i})$$

$$Q_j = S_Q(u_{j1}) = S_Q(u_{j2}) = \cdots = S_Q(u_{jt_j})$$

Thus,

$$|P_i| = |S_P(u_{i1})| = |S_P(u_{i2})| = \cdots = |S_P(u_{is_i})|$$

$$|Q_j| = |S_Q(u_{j1})| = |S_Q(u_{j2})| = \cdots = |S_Q(u_{jt_j})|$$

If $P \triangleright Q$, from Definition 4.1, it follows that there exists a sequence $K'(Q)$ of $K(Q)$ such that for $\forall i \in \{1, 2, \ldots, |U|\}$, $|S_P(u_i)| \leq |S_Q(u'_i)|$ and $|S_P(u_{i_0})| < |S_Q(u'_{i_0})|$ for some $u_{i_0} \in U$.

Hence

$$\begin{aligned}GK(P) &= \frac{1}{|U|^2}\sum_{i=1}^{m}|P_i|^2\\ &= \frac{1}{|U|^2}\sum_{i=1}^{m}(|S_P(u_{i1})| + |S_P(u_{i2})|\\ &\quad + \cdots + |S_P(u_{is_i})|)\\ &= \frac{1}{|U|^2}(|S_P(u_1)| + |S_P(u_2)|\\ &\quad + \cdots + |S_P(u_{|U|})|)\\ &= \frac{1}{|U|^2}\left(\sum_{i=1,i\neq i_0}^{|U|}|S_P(u_i)| + |S_P(u_{i_0})|\right)\\ &< \frac{1}{|U|^2}\left(\sum_{i=1,i\neq i_0}^{|U|}|S_Q(u_i')| + |S_Q(u_{i_0}')|\right)\\ &= \frac{1}{|U|^2}(|S_Q(u_1')| + |S_Q(u_2')| + \cdots + |S_Q(u_U')|)\\ &= \frac{1}{|U|^2}(|S_Q(u_1)| + |S_Q(u_2)| + \cdots + |S_Q(u_{|U|})|)\\ &= \frac{1}{|U|^2}\sum_{j=1}^{n}(|S_Q(u_{j1})| + |S_Q(u_{j2})|\\ &\quad + \cdots + |S_Q(u_{jt_j})|)\\ &= \frac{1}{|U|^2}\sum_{j=1}^{n}|Q_j|^2 = GK(Q)\end{aligned}$$

i.e., $GK(P) < GK(Q)$.

This completes this proof. ■

Definition 5.2 (Liang *et al.*, 2006) Let $S=(U,A)$be an incomplete information system, $K(A)=\{S_A(u_1), S_A(u_2), \ldots, S_A(u_{|U|})\}$. An information granulation of A is defined by

$$GK(A) = \frac{1}{|U|^2}\sum_{i=1}^{m}|S_A(u_i)|$$

where $\frac{1}{|U|} \leq GK(A) \leq 1$.

Theorem 5.2 Let $S=(U,A)$be an incomplete information system, $P,Q \subseteq A$. If $P \triangleright Q$, then $GK(P) < GK(Q)$.

Proof Similar to the proof of Theorem 5.1, it can be easily proved. ■

Definition 5.3 (Qian & Liang, 2008) Let $S=(U,A)$ be a complete information system, $U/IND(A)=\{P_1, P_2, \ldots, P_m\}$. Combination granulation of A is defined by

$$CG(A) = \sum_{i=1}^{m}\frac{|P_i|}{|U|}\frac{C_{|P_i|}^2}{C_{|U|}^2}$$

where $0 \leq CG(A) \leq 1$; $\frac{|P_i|}{|U|}$ represents the probability of equivalence class P_i within the universe U and $\frac{C_{|P_i|}^2}{C_{|U|}^2}$ denotes the probability of pairs of elements on equivalence class P_i within the whole pairs of elements on the universe U.

Theorem 5.3 Let $S=(U,A)$ be a complete information system, $P,Q \subseteq A$. If $P \triangleright Q$, then $CG(P) < CG(Q)$.

Proof Let $P,Q \subseteq A$. For a complete information system, $U/IND(P)=\{P_1, P_2, \ldots, P_m\}$, $P_i=\{u_{i1}, u_{i2}, \ldots, u_{is_i}\}$, where $|P_i|=s_i$ and $\sum_{i=1}^{m} s_i=|U|$; $U/IND(Q)=\{Q_1, Q_2, \ldots, Q_n\}$, $Q_j=\{u_{j1}, u_{j2}, \ldots, u_{jt_j}\}$, where $|Q_j|=t_j$ and $\sum_{j=1}^{n} t_j=|U|$. Suppose that $K(P)=\{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$ and $K(Q)=\{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$ in incomplete information systems.

Therefore, like the proof of Theorem 5.1, we have that

$$|P_i|=|S_P(u_{i1})|=|S_P(u_{i2})|= \cdots =|S_P(u_{is_i})|,$$

$$|Q_j|=|S_Q(u_{j1})|=|S_Q(u_{j2})|= \cdots =|S_Q(u_{jt_j})|$$

If $P \triangleright Q$, from Definition 4.1, it follows that there exists a sequence $K'(Q)$ of $K(Q)$ such that for $\forall i \in \{1, 2, \ldots, |U|\}$, $|S_P(u_i)| \leq |S_Q(u_i')|$ and $|S_P(u_{i_0})| < |S_Q(u_{i_0}')|$ for some $u_{i_0} \in U$.

Hence,

$$\begin{aligned}CG(P) &= \sum_{i=1}^{m}\frac{|P_i|}{|U|}\frac{C_{|P_i|}^2}{C_{|U|}^2}\\ &= \frac{1}{|U|}\sum_{i=1}^{m}\frac{C_{|P_i|}^2}{C_{|U|}^2}\left(\frac{|S_P(u_{i1})|}{|P_i|} + \frac{|S_P(u_{i2})|}{|P_i|}\right.\\ &\quad \left. + \cdots + \frac{|S_P(u_{is_i})|}{|P_i|}\right)\\ &= \frac{1}{|U|C_{|U|}^2}(C_{|S_P(u_1)|}^2 + C_{|S_P(u_2)|}^2\\ &\quad + \cdots + C_{|S_P(u_{|U|})|}^2)\end{aligned}$$

$$= \frac{1}{|U|}\left(\sum_{i=1,i\neq i_0}^{|U|}\frac{C^2_{|S_P(u_i)|}}{C^2_{|U|}}+\frac{C^2_{|S_P(u_{i_0})|}}{C^2_{|U|}}\right)$$

$$< \frac{1}{|U|}\left(\sum_{i=1,i\neq i_0}^{|U|}\frac{C^2_{|S_Q(u'_i)|}}{C^2_{|U|}}+\frac{C^2_{|S_Q(u'_{i_0})|}}{C^2_{|U|}}\right)$$

$$= \frac{1}{|U|C^2_{|U|}}\left(C^2_{|S_Q(u'_1)|}+C^2_{|S_Q(u'_2)|}+\cdots+C^2_{|S_Q(u'_{|U|})|}\right)$$

$$= \frac{1}{|U|C^2_{|U|}}\left(C^2_{|S_Q(u_1)|}+C^2_{|S_Q(u_2)|}+\cdots+C^2_{|S_Q(u_{|U|})|}\right)$$

$$= \frac{1}{|U|}\sum_{j=1}^{n}\frac{C^2_{|Q_j|}}{C^2_{|U|}}\left(\frac{|S_Q(u_{j1})|}{|Q_j|}+\frac{|S_Q(u_{j2})|}{|Q_j|} + \cdots + \frac{|S_Q(u_{jt_j})|}{|Q_j|}\right)$$

$$= \sum_{j=1}^{n}\frac{|Q_j|}{|U|}\frac{C^2_{|Q_j|}}{C^2_{|U|}}$$

$$= CG(Q)$$

i.e., $CG(P) < CG(Q)$.

This completes this proof. ■

Definition 5.4 (Qian & Liang, 2008) Let $S=(U,A)$be an incomplete information system, $K(A)=\{S_A(u_1), S_A(u_2), \ldots, S_A(u_{|U|})\}$. Combination granulation of A is defined by

$$CG(A)=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C^2_{|S_A(u_i)|}}{C^2_{|U|}}$$

where $0\leq CG(A)\leq 1$ and $\frac{C^2_{|S_A(u_i)|}}{C^2_{|U|}}$ denotes the probability of pairs of elements on tolerance class $S_A(u_i)$ within the whole pairs of elements on the universe U.

Theorem 5.4 Let $S=(U,A)$ be an incomplete information system and $P,Q\subseteq A$. If $P \rhd Q$, then $CG(P) < CG(Q)$.

Proof Similar to the proof of Theorem 5.3, it can be easily proved. ■

The concept of rough entropy was introduced to measure the roughness of knowledge in rough set theory (Liang & Xu, 2002; Liang & Shi, 2004; Liang & Li, 2005; Liang *et al.*, 2006). Essentially, it is also a form of information granulation in information systems.

Definition 5.5 (Liang & Shi, 2004) Let $S=(U,A)$ be a complete information system, $U/IND(A)=\{P_1, P_2, \ldots, P_m\}$. Rough entropy of A is defined by

$$E_r(A) = -\sum_{i=1}^{m}\frac{|P_i|}{|U|}\log_2\frac{1}{|P_i|}$$

where $0\leq E_r(A)\leq \log_2|U|$.

Theorem 5.5 Let $S=(U,A)$ be a complete information system and $P,Q\subseteq A$. If $P\rhd Q$, then $E_r(P)\leq E_r(Q)$.

Proof Let $P,Q\subseteq A$. In complete information systems, $U/IND(P)=\{P_1,P_2,\ldots,P_m\}$, $P_i=\{u_{i1},u_{i2},\ldots,u_{is_i}\}$, where $|P_i|=s_i$ and $\sum_{i=1}^{m} s_i=|U|$; $U/IND(Q)=\{Q_1,Q_2,\ldots,Q_n\}$, $Q_j=\{u_{j1},u_{j2},\ldots,u_{jt_j}\}$, where $|Q_j|=t_j$ and $\sum_{j=1}^{n} t_j=|U|$. Suppose that $K(P)=\{S_P(u_1), S_P(u_2),\ldots,S_P(u_{|U|})\}$, and $K(Q)=\{S_Q(u_1), S_Q(u_2),\ldots,S_Q(u_{|U|})\}$ in incomplete information systems.

Therefore, like the proof of Theorem 5.1, we have that

$$|P_i|=|S_P(u_{i1})|=|S_P(u_{i2})|=\cdots=|S_P(u_{is_i})|$$
$$|Q_j|=|S_Q(u_{j1})|=|S_Q(u_{j2})|=\cdots=|S_Q(u_{jt_j})|$$

If $P\rhd Q$, from Definition 4.1, it follows that there exists a sequence $K'(Q)$of $K(Q)$ such that for $\forall i\in\{1,2,\ldots,|U|\}$, $|S_P(u_i)|\leq|S_Q(u'_i)|$ and $|S_P(u_{i_0})| < |S_Q(u'_{i_0})|$ for some $u_{i_0}\in U$.

Hence,

$$E_r(P) = -\sum_{i=1}^{m}\frac{|P_i|}{|U|}\log_2\frac{1}{|P_i|}$$

$$= -\sum_{i=1}^{m}\left(\frac{1}{|U|}\log_2\frac{1}{|S_P(u_{i1})|} + \frac{1}{|U|}\log_2\frac{1}{|S_P(u_{i2})|}+\cdots+\frac{1}{|U|}\log_2\frac{1}{|S_P(u_{is_i})|}\right)$$

$$= -\left(\frac{1}{|U|}\log_2\frac{1}{|S_P(u_1)|} + \frac{1}{|U|}\log_2\frac{1}{|S_P(u_2)|} + \cdots + \frac{1}{|U|}\log_2\frac{1}{|S_P(u_{|U|})|}\right)$$

$$= \frac{1}{|U|}\sum_{i=1}^{|U|}\log_2|S_P(u_i)|$$

$$= \frac{1}{|U|}\log_2\prod_{i=1,i\neq i_0}^{|U|}|S_P(u_i)| + \frac{1}{|U|}\log_2|S_P(u_{i_0})|$$

$$< \frac{1}{|U|}\log_2\prod_{i=1,i\neq i_0}^{|U|}|S_Q(u'_i)| + \frac{1}{|U|}\log_2|S_Q(u'_{i_0})|$$

$$= -\left(\frac{1}{|U|}\log_2\frac{1}{|S_Q(u'_1)|} + \frac{1}{|U|}\log_2\frac{1}{|S_Q(u'_2)|} + \cdots + \frac{1}{|U|}\log_2\frac{1}{|S_Q(u'_{|U|})|}\right)$$

$$= -\sum_{j=1}^{n}\left(\frac{1}{|U|}\log_2\frac{1}{|S_Q(u_1)|} + \frac{1}{|U|}\log_2\frac{1}{|S_Q(u_2)|} + \cdots + \frac{1}{|U|}\log_2\frac{1}{|S_Q(u_{|U|})|}\right)$$

$$= -\sum_{j=1}^{n}\frac{|Q_j|}{|U|}\log_2\frac{1}{|Q_j|}$$

$$= E_r(Q)$$

i.e., $E_r(P) < E_r(Q)$.

This completes this proof. ■

Definition 5.6 (Liang *et al.*, 2006) Let $S=(U,A)$ be an incomplete information system and $K(A)=\{S_A(u_1), S_A(u_2), \ldots, S_A(u_{|U|})\}$. Rough entropy of A is defined by

$$E_r(A) = -\sum_{i=1}^{|U|}\frac{1}{|U|}\log_2\frac{1}{|S_A(u_i)|}$$

where $0 \leq E_r(A) \leq \log_2|U|$.

Theorem 5.6 Let $S=(U,A)$ be an incomplete information system and $P, Q \subseteq A$. If $P \triangleright Q$, then $E_r(P) < E_r(Q)$.

Proof It can be easily proved according to the idea of Theorem 5.5. ■

Remark. The above theorems show that these forms of information granulation satisfy both rough monotonicity and granulation monotonicity. We can draw a conclusion from these results that information granulation must satisfy rough monotonicity if it satisfies granulation monotonicity. However, its reverse relation cannot be established in general. Hence, partial ordering $\trianglerighteq$ might be better to characterize the nature of information granulation than partial ordering $\preceq$ in information systems.

6. Granulation monotonicities of several existing information entropies

6.1. *Granulation monotonicity of Shannon's entropy*

The entropy of a system as defined by Shannon (Shannon, 1948) can also be used to measure the uncertainty of an information system. In Shannon's entropy, an equivalence partition is regarded as a finite probability distribution, and the proportion of each equivalence class from a given partition within the universe is seen as its probability on the universe. It can be formally defined as follows:

Definition 6.1 (Shannon, 1948) Let $S=(U,A)$ be a complete information system, $U/IND(A)=\{X_1, X_2, \ldots, X_m\}$. Shannon's entropy of A is defined as

$$H(A) = -\sum_{i=1}^{m}p_i\log_2 p_i = -\sum_{i=1}^{m}\frac{|X_i|}{|U|}\log_2\frac{|X_i|}{|U|}$$

where $p_i = \frac{|X_i|}{|U|}$ represents the probability of equivalence class X_i within the universe U.

The following theorem gives the rough monotonicity of Shannon's entropy in information systems.

Theorem 6.1 (Rough monotonicity) Let $S=(U,A)$ be a complete information system and $P, Q \subseteq A$. Then, $H(P) \geq H(Q)$ if $P \preceq Q$.

Proof Let $S=(U,A)$ be a complete information system, $P, Q \subseteq A$ with $P \preceq Q$, $U/IND(P)=\{P_1, P_2, \ldots, P_m\}$ and $U/IND(Q)=\{Q_1, Q_2,$

$\ldots, Q_n\}$. In the context of incomplete information systems, they can also be written as $P_i = \{u_{i1}, u_{i2}, \ldots, u_{is_i}\}$, where $|P_i| = s_i$ and $\sum_{i=1}^{m} s_i = |U|$; and $U/SIM(Q) = \{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$, $Q_j = \{u_{j1}, u_{j2}, \ldots, u_{js_j}\}$, where $|Q_j| = s_j$ and $\sum_{j=1}^{n} s_j = |U|$. Hence, $P_i = S_P(u_{i1}) = S_P(u_{i2}) = \cdots = S_P(u_{is_i})$ and $Q_j = S_Q(u_{j1}) = S_Q(u_{j2}) = \cdots = S_Q(u_{js_j})$.

Since $P \preceq Q$, we have that $S_P(u_i) \subseteq S_Q(u_i)$, $\forall i \in \{1, 2, \ldots, |U|\}$. Therefore, we obtain that

$$
\begin{aligned}
H(P) &= -\sum_{i=1}^{m} \frac{|P_i|}{|U|} \log_2 \frac{|P_i|}{|U|} \\
&= -\sum_{i=1}^{m} \frac{1}{|U|} \left(\log_2 . \frac{|S_P(u_{i1})|}{|U|} + \log_2 \frac{|S_P(u_{i2})|}{|U|} \right. \\
&\quad \left. + \cdots + \log_2 \frac{|S_P(u_{is_i})|}{|U|} \right) \\
&= -\frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 \frac{|S_P(u_i)|}{|U|} \\
&\geq -\frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u_i)|}{|U|} \\
&= -\sum_{i=1}^{n} \frac{1}{|U|} \left(\log_2 \frac{|S_Q(u_{j1})|}{|U|} + \log_2 \frac{|S_Q(u_{j2})|}{|U|} \right. \\
&\quad \left. + \cdots + \log_2 \frac{|S_Q(u_{js_j})|}{|U|} \right) \\
&= -\sum_{j=1}^{n} \frac{|Q_j|}{|U|} \log_2 \frac{|Q_j|}{|U|} = H(Q)
\end{aligned}
$$

i.e., $H(P) \geq H(Q)$. This completes the proof. ■

Theorem 6.1 suggests that Shannon's entropy H of a complete information system increases as equivalence classes become smaller through finer classification.

Using the partial relation $\trianglerighteq$, we suggest granulation monotonicity of Shannon's entropy in information systems.

Theorem 6.2 (Granulation monotonicity) Let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. Then, $H(P) \geq H(Q)$ if $P \trianglerighteq Q$.

Proof Similar to Theorem 6.1, it follows from $P \trianglerighteq Q$ that there exists a sequence $\{S_Q(u'_1), S_Q(u'_2), \ldots, S_Q(u'_{|U|})\}$ such that $|S_P(u_i)| \leq |S_Q(u'_i)|$, $\forall i \in \{1, 2, \ldots, |U|\}$. Hence, we have that

$$
\begin{aligned}
H(P) &= -\sum_{i=1}^{m} \frac{|P_i|}{|U|} \log_2 \frac{|P_i|}{|U|} \\
&= -\frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 \frac{|S_P(u_i)|}{|U|} \\
&\geq -\frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u'_i)|}{|U|} \\
&= -\frac{1}{|U|} \sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u_i)|}{|U|} \\
&= -\sum_{i=1}^{n} \frac{1}{|U|} \left(\log_2 \frac{|S_Q(u_{j1})|}{|U|} + \log_2 \frac{|S_Q(u_{j2})|}{|U|} \right. \\
&\quad \left. + \cdots + \log_2 \frac{|S_Q(u_{js_j})|}{|U|} \right) \\
&= -\sum_{j=1}^{n} \frac{|Q_j|}{|U|} \log_2 \frac{|Q_j|}{|U|} = H(Q)
\end{aligned}
$$

i.e., $H(P) \geq H(Q)$. This completes the proof. ■

Theorem 6.2 states that Shannon's entropy H of a complete information system increases as the information granularity of this system becomes finer.

Corollary 6.1 let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. Then, $H(P) < H(Q)$ if $G(P) > G(Q)$.

In original Shannon's information entropy, the proportion of each equivalence class within the universe is regarded as its probability. Clearly, it is difficult to measure the uncertainty of an incomplete information system. To overcome this limitation, Liang *et al.* (2006) extended Shannon's entropy to incomplete information systems, which is as follows:

Definition 6.2 Let $S = (U, A)$ be an incomplete information system and $U/SIM(A) = \{S_A(u_1), S_A(u_2), \ldots, S_A(u_{|U|})\}$. Information entropy of A

is defined as

$$H^*(A) = -\sum_{i=1}^{|U|} \frac{1}{|U|} \log_2 \frac{|S_A(u)|}{|U|}$$

where $H^* : A \to [0, \infty)$.

In the following, we obtain the rough monotonicity and the granulation monotonicity of the extended Shannon's entropy in incomplete information systems.

Theorem 6.3 (Rough monotonicity) Let $S=(U, A)$ be an incomplete information system and $P, Q \subseteq A$. Then, $H^*(P) \geq H^*(Q)$ if $P \preceq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$. Since $P \preceq Q$, we have that $S_P(u_i) \subseteq S_Q(u_i)$ and $|S_P(u_i)| \leq |S_Q(u_i)|$, $\forall i \in \{1, 2, \ldots, |U|\}$. Therefore, one has that

$$\begin{aligned} H^*(P) &= -\frac{1}{|U|}\sum_{i=1}^{|U|} \log_2 \frac{|S_P(u_i)|}{|U|} \\ &\geq -\frac{1}{|U|}\sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u_i)|}{|U|} = H^*(Q) \end{aligned}$$

i.e., $H^*(P) \geq H^*(Q)$. This completes the proof. ■

Theorem 6.3 suggests that the extended Shannon's entropy H^* of an incomplete information system increases as tolerance classes become smaller through finer classification.

The following theorem gives the granulation monotonicity of incomplete Shannon's entropy H^* in incomplete information systems.

Theorem 6.4 (Granulation monotonicity) Let $S=(U, A)$ be an incomplete information system and $P, Q \subseteq A$. Then, $H^*(P) \geq H^*(Q)$ if $P \trianglerighteq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$. From the definition of $P \trianglerighteq Q$, it follows that there exists a sequence $\{S_Q(u'_1), S_Q(u'_2), \ldots, S_Q(u'_{|U|})\}$ such that $|S_P(u_i)| \leq |S_Q(u'_i)|$, $\forall i \in \{1, 2, \ldots, |U|\}$. Hence, we have that

$$\begin{aligned} H(P) &= -\frac{1}{|U|}\sum_{i=1}^{|U|} \log_2 \frac{|S_P(u_i)|}{|U|} \\ &\geq -\frac{1}{|U|}\sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u'_i)|}{|U|} \\ &= -\frac{1}{|U|}\sum_{i=1}^{|U|} \log_2 \frac{|S_Q(u_i)|}{|U|} = H(Q) \end{aligned}$$

i.e., $H(P) \geq H(Q)$. This completes the proof. ■

Theorem 6.4 states that the extended Shannon's entropy H^* of an incomplete information system increases as the information granularity of this system becomes finer.

Corollary 6.2 let $S=(U, A)$ be a complete information system and $P, Q \subseteq A$. Then, $H^*(P) < H^*(Q)$ if $G(P) > G(Q)$.

6.2. Granulation monotonicity of Liang's entropy

Although several authors have used Shannon's entropy and its variants to measure uncertainty in information systems, it has some limitations. In fact, Shannon's entropy is not a fuzzy entropy, and cannot measure the fuzziness in information systems. To overcome the limitation, Liang *et al.* (2002) proposed a new information entropy. Unlike the logarithmic behaviour of Shannon's entropy, the gain function of this entropy possesses a complement nature. The new entropy can be used to measure both the uncertainty of an information system and the fuzziness of a rough set and a rough classification in the rough set theory. In complete information systems, Liang's information entropy is defined by the following.

Definition 6.3 let$S=(U, A)$ be a complete information system and$U/IND(A)=\{X_1, X_2, \ldots, X_m\}$. The information entropy of A is defined as

$$IE(A) = \sum_{i=1}^{m} \frac{|X_i|}{|U|}\frac{|X_i^c|}{|U|}$$

where X_i^c is the complement set of X_i, i.e., $X_i^c = U - X_i$.

Like Shannon's entropy, one can obtain the rough monotonicity and granulation monotonicity of the above information entropy *IE*, which are listed in the following Theorems 18 and 19.

Theorem 6.5 (Rough monotonicity) Let $S=(U,A)$ be a complete information system and $P,Q \subseteq A$. Then, $IE(P) \geq IE(Q)$ if $P \preceq Q$.

Proof Let $U/IND(P)=\{P_1,P_2,\ldots,P_m\}$ and $U/IND(Q)=\{Q_1,Q_2,\ldots,Q_n\}$. In the context of incomplete information systems, they can be written as $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$, $P_i=\{u_{i1},u_{i2},\ldots,u_{is_i}\}$, where $|P_i|=s_i$ and $\sum_{i=1}^{m} s_i=|U|$; and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$, $Q_j=\{u_{j1},u_{j2},\ldots,u_{js_j}\}$, where $|Q_j|=s_j$ and $\sum_{j=1}^{n} s_i=|U|$. Hence, $P_i=S_P(u_{i1})=S_P(u_{i2})=\cdots=S_P(u_{is_i})$ and $Q_j=S_Q(u_{j1})=S_Q(u_{j2})=\cdots=S_Q(u_{js_j})$.

It follows from $P \preceq Q$ that $S_P(u_i) \subseteq S_Q(u_i)$ and $|S_P(u_i)| \leq |S_Q(u_i)|$, $\forall i \in \{1,2,\ldots,|U|\}$. Hence, we have that

$$
\begin{aligned}
IE(P) &= \sum_{i=1}^{m} \frac{|P_i|}{|U|}\frac{|P_i^c|}{|U|} = \sum_{i=1}^{m} \frac{|P_i|}{|U|}\left(1-\frac{|P_i|}{|U|}\right) \\
&= \sum_{i=1}^{m}\left(\frac{1}{|U|}\left(1-\frac{|S_P(u_{i1})|}{|U|}\right)+\frac{1}{|U|}\left(1-\frac{|S_P(u_{i2})|}{|U|}\right)\right. \\
&\quad \left. + \cdots + \frac{1}{|U|}\left(1-\frac{|S_P(u_{is_i})|}{|U|}\right)\right) \\
&= \frac{1}{|U|}\left(1-\frac{|S_P(u_1)|}{|U|}\right)+\frac{1}{|U|}\left(1-\frac{|S_P(u_2)|}{|U|}\right) \\
&\quad + \cdots + \frac{1}{|U|}\left(1-\frac{|S_P(u_{|U|})|}{|U|}\right) \\
&\geq \frac{1}{|U|}\left(1-\frac{|S_Q(u_1)|}{|U|}\right)+\frac{1}{|U|}\left(1-\frac{|S_Q(u_2)|}{|U|}\right) \\
&\quad + \cdots + \frac{1}{|U|}\left(1-\frac{|S_Q(u_{|U|})|}{|U|}\right) \\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u_i)|}{|U|}\right) = \sum_{j=1}^{n}\frac{|Q_j|}{|U|}\frac{|Q_j^c|}{|U|} = IE(Q)
\end{aligned}
$$

i.e., $IE(P) \geq IE(Q)$. This completes the proof. ■

Theorem 6.5 shows that the information entropy IE of a complete information system increases as equivalence classes become smaller through much finer classification.

Theorem 6.6 (Granulation monotonicity) Let $S=(U,A)$ be a complete information system and $P,Q \subseteq A$. Then, $IE(P) \geq IE(Q)$ if $P \trianglerighteq Q$.

Proof Like Theorem 6.5, one can suppose that $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$. Since $P \trianglerighteq Q$, it easily follows that there exists a sequence $\{S_Q(u_1'),S_Q(u_2'),\ldots,S_Q(u_{|U|}')\}$ such that $|S_P(u_i)| \leq |S_Q(u_i')|$, $\forall i \in \{1,2,\ldots,|U|\}$. Hence, we have that

$$
\begin{aligned}
IE(P) &= \sum_{i=1}^{m} \frac{|P_i|}{|U|}\frac{|P_i^c|}{|U|} = \sum_{i=1}^{m} \frac{|P_i|}{|U|}\left(1-\frac{|P_i|}{|U|}\right) \\
&= \sum_{i=1}^{m}\left(\frac{1}{|U|}\left(1-\frac{|S_P(u_{i1})|}{|U|}\right)+\frac{1}{|U|}\left(1-\frac{|S_P(u_{i2})|}{|U|}\right)\right. \\
&\quad \left. + \cdots + \frac{1}{|U|}\left(1-\frac{|S_P(u_{is_i})|}{|U|}\right)\right) \\
&= \frac{1}{|U|}\left(1-\frac{|S_P(u_1)|}{|U|}\right)+\frac{1}{|U|}\left(1-\frac{|S_P(u_2)|}{|U|}\right) \\
&\quad + \cdots + \frac{1}{|U|}\left(1-\frac{|S_P(u_{|U|})|}{|U|}\right) \\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_P(u_i)|}{|U|}\right) \\
&\geq \sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u_i')|}{|U|}\right) \\
&= \sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u_i)|}{|U|}\right) \\
&= \sum_{j=1}^{n}\frac{|Q_j|}{|U|}\frac{|Q_j^c|}{|U|} = IE(Q)
\end{aligned}
$$

that is $IE(P) \geq IE(Q)$. This completes the proof. ■

Theorem 6.6 states that information entropy IE of a complete information system increases as the information granularity of this system becomes finer.

Corollary 6.3 Let $S=(U,A)$be a complete information system and $P,Q \subseteq A$. Then, $IE(P) < IE(Q)$ if $G(P) > G(Q)$.

Similar to Shannon's entropy, Liang's information entropy IE also encounters the same challenge for dealing with incomplete data. Liang *et al.* (2006) gave the definition of information entropy IE in incomplete information systems, which is shown as follows:

Definition 6.4 Let $S=(U,A)$ be an incomplete information system and $U/SIM(A)=\{S_A(u_1), S_A(u_2), \ldots, S_A(u_{|U|})\}$. The information entropy of A is defined as

$$IE^*(A)=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_A(u_i)|}{|U|}\right)$$

In the sequel, we examine the rough monotonicity and the granulation monotonicity of the information entropy IE^* in incomplete information systems.

Theorem 6.7 (Rough monotonicity) Let $S=(U,A)$ be an incomplete information system and $P,Q\subseteq A$. Then, $IE^*(P)\geq IE^*(Q)$ if $P\preceq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots, S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots, S_Q(u_{|U|})\}$. It follows from $P\preceq Q$ that $S_P(u_i)\subseteq S_Q(u_i)$ and $|S_P(u_i)|\leq|S_Q(u_i)|$, $\forall i\in\{1,2,\ldots,|U|\}$. Hence, we have that

$$\begin{aligned}IE^*(P)&=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_A(u_i)|}{|U|}\right)\\&\geq\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u_i)|}{|U|}\right)\\&=IE^*(Q)\end{aligned}$$

i.e., $IE^*(P)\geq IE^*(Q)$. This completes the proof. ■

Theorem 6.7 suggests that the information entropy IE^* of an incomplete information system increases as tolerance classes become smaller through finer classification.

Theorem 6.8 (Granulation monotonicity) Let $S=(U,A)$ be an incomplete information system and $P,Q\subseteq A$. Then, $IE^*(P)\geq IE^*(Q)$ if $P\trianglerighteq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots, S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots, S_Q(u_{|U|})\}$. Since $P\trianglerighteq Q$, it easily follows that there exists a sequence $\{S_Q(u'_1),S_Q(u'_2),\ldots, S_Q(u'_{|U|})\}$ such that $|S_P(u_i)|\leq|S_Q(u'_i)|$, $\forall i\in\{1,2,\ldots,|U|\}$. Thus, one has that

$$\begin{aligned}IE^*(P)&=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_A(u_i)|}{|U|}\right)\\&\geq\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u'_i)|}{|U|}\right)\\&=\sum_{i=1}^{|U|}\frac{1}{|U|}\left(1-\frac{|S_Q(u_i)|}{|U|}\right)\\&=IE^*(Q)\end{aligned}$$

that is $IE^*(P)\geq IE^*(Q)$. This completes the proof. ■

Theorem 6.8 states that information entropy IE^* of an incomplete information system increases as the information granularity of this system becomes finer.

Corollary 6.4 Let $S=(U,A)$ be a complete information system and $P,Q\subseteq A$. Then, $IE^*(P)<IE^*(Q)$ if $G(P)>G(Q)$.

6.3. Granulation monotonicity of combination entropy

From the viewpoint of the rough set theory, knowledge is a kind of classification ability (Pawlak, 1991). In general, the elements in an equivalence class cannot be distinguished from each other, but the elements in different equivalence classes can be distinguished from each other. However, both Shannon's entropy and Liang's information entropy cannot reflect the behaviour that any two elements that can be distinguished from each other are regarded as a knowledge unit on the universe. Based on the consideration, Qian and Liang (2008) presented a new measure, called combination entropy, for measuring the uncertainty and information content of an information system, in which the information content is depicted by the number of pairs of the objects that can be distinguished from each other on the universe. Definition 6.5 gives the description of the combination entropy.

Definition 6.5 Let $S=(U,A)$ be a complete information system and $U/IND(A)=\{X_1,X_2,$

$\ldots, X_m\}$. Combination entropy of A is defined as

$$CE(A) = \sum_{i=1}^{m} \frac{|X_i|}{|U|} \frac{C_{|U|}^2 - C_{|X_i|}^2}{C_{|U|}^2}$$

where $C_{|X_i|}^2 = \frac{|X_i| \times (|X_i|-1)}{2}$, $\frac{|X_i|}{|U|}$ represents the probability of an equivalence class X_i within the universe U, and $\frac{C_{|U|}^2 - C_{|X_i|}^2}{C_{|U|}^2}$ denotes the probability of pairs of the elements that are distinguishable from each other within the entire number of pairs of the elements in the universe.

The following theorem shows the rough monotonicity of the combination entropy.

Theorem 6.9 (Rough monotonicity) Let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. Then, $CE(P) \geq CE(Q)$ if $P \preceq Q$.

Proof Let $U/IND(P) = \{P_1, P_2, \ldots, P_m\}$ and $U/IND(Q) = \{Q_1, Q_2, \ldots, Q_n\}$. In the context of incomplete information systems, they can be written as $U/SIM(P) = \{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$, $P_i = \{u_{i1}, u_{i2}, \ldots, u_{is_i}\}$, where $|P_i| = s_i$ and $\sum_{i=1}^{m} s_i = |U|$; and $U/SIM(Q) = \{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$, $Q_j = \{u_{j1}, u_{j2}, \ldots, u_{js_j}\}$ where $|Q_j| = s_j$ and $\sum_{j=1}^{n} s_j = |U|$. Hence, $P_i = S_P(u_{i1}) = S_P(u_{i2}) = \cdots = S_P(u_{is_i})$ and $Q_j = S_Q(u_{j1}) = S_Q(u_{j2}) = \cdots = S_Q(u_{js_j})$. It follows from $P \preceq Q$ that $S_P(u_i) \subseteq S_Q(u_i)$ and $|S_P(u_i)| \leq |S_Q(u_i)|$, $\forall i \in \{1, 2, \ldots, |U|\}$. Hence, we have that

$$\begin{aligned} CE(P) &= \sum_{i=1}^{m} \frac{|X_i|}{|U|} \frac{C_{|U|}^2 - C_{|X_i|}^2}{C_{|U|}^2} = 1 - \frac{1}{|U|}\sum_{i=1}^{m} |X_i| \times \frac{C_{|X_i|}^2}{C_{|U|}^2} \\ &= 1 - \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2} \geq 1 - \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2} \\ &= \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|U|}^2 - C_{|S_Q(u_i)|}^2}{C_{|U|}^2} = CE(Q) \end{aligned}$$

i.e., $CE(P) \geq CE(Q)$. This completes the proof. ■

Theorem 6.9 shows that the combination entropy CE of a complete information system increases with equivalence classes becoming smaller through finer classification.

Using the partial relation $\preceq'$, one can obtain the granulation monotonicity of the combination entropy in a complete information system.

Theorem 6.10 (Granulation monotonicity) Let $S = (U, A)$ be a complete information system and $P, Q \subseteq A$. Then, $CE(P) \geq CE(Q)$ if $P \trianglerighteq Q$.

Proof Using the denotations in Theorem 6.9, we denoted two equivalence partitions by $U/SIM(P) = \{S_P(u_1), S_P(u_2), \ldots, S_P(u_{|U|})\}$ and $U/SIM(Q) = \{S_Q(u_1), S_Q(u_2), \ldots, S_Q(u_{|U|})\}$. Since $P \trianglerighteq Q$, it easily follows that there exists a sequence $\{S_Q(u_1'), S_Q(u_2'), \ldots, S_Q(u_{|U|}')\}$ such that $|S_P(u_i)| \leq |S_Q(u_i')|$, $\forall i \in \{1, 2, \ldots, |U|\}$. Thus, one has that

$$\begin{aligned} CE(P) &= \sum_{i=1}^{m} \frac{|X_i|}{|U|} \frac{C_{|U|}^2 - C_{|X_i|}^2}{C_{|U|}^2} \\ &= 1 - \frac{1}{|U|}\sum_{i=1}^{m} |X_i| \times \frac{C_{|X_i|}^2}{C_{|U|}^2} = 1 - \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2} \\ &\geq 1 - \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|S_Q(u_i')|}^2}{C_{|U|}^2} = 1 - \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2} \\ &= \frac{1}{|U|}\sum_{i=1}^{|U|} \frac{C_{|U|}^2 - C_{|S_Q(u_i)|}^2}{C_{|U|}^2} = CE(Q) \end{aligned}$$

i.e., $CE(P) \geq CE(Q)$. This completes the proof. ■

Theorem 6.10 states that the combination entropy CE of a complete information system increases with the information granularity of this system becoming finer.

Corollary 6.5 Let $S = (U, A)$ be an incomplete information system and $P, Q \subseteq A$. Then, $CE(P) < CE(Q)$ if $G(P) > G(Q)$.

In an incomplete information system, the elements in a tolerance class may not be distinguished from each other, but the elements in different tolerance classes are probably distinguishable from each other. Based on this consideration, Qian *et al.* (2009) proposed an incomplete combination entropy in incomplete information systems. The following definition gives the description of the incomplete combination entropy.

Definition 6.6 Let $S=(U,A)$ be an incomplete information system and $U/SIM(A)=\{S_A(u_1),S_A(u_2),\ldots,S_A(u_{|U|})\}$. The combination entropy of A is defined as

$$CE^*(A)=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|U|}^2-C_{|S_A(u_i)|}^2}{C_{|U|}^2}$$

where $\frac{C_{|U|}^2-C_{|S_A(u_i)|}^2}{C_{|U|}^2}$ denotes the probability of pairs of elements that are probably distinguishable from each other within the whole number of pairs of elements on the universe.

Theorem 6.11 (Rough monotonicity) Let $S=(U,A)$ be an incomplete information system and $P,Q\subseteq A$. Then, $CE^*(P)\geq CE^*(Q)$ if $P\preceq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$. It follows from $P\preceq Q$ that $S_P(u_i)\subseteq S_Q(u_i)$ and $|S_P(u_i)|\leq|S_Q(u_i)|$, $\forall i\in\{1,2,\ldots,|U|\}$. Hence, we have that

$$\begin{aligned}CE^*(P)&=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|U|}^2-C_{|S_P(u_i)|}^2}{C_{|U|}^2}\\&=1-\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2}\geq1-\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2}\\&=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|U|}^2-C_{|S_Q(u_i)|}^2}{C_{|U|}^2}=CE^*(Q)\end{aligned}$$

i.e., $CE^*(P)\geq CE^*(Q)$. This completes the proof. ■

Theorem 6.11 suggests that the combination entropy CE_* of an incomplete information system increases as the tolerance classes become smaller through finer classification.

Theorem 6.12 (Granulation monotonicity) Let $S=(U,A)$be an incomplete information system and$P,Q\subseteq A$. Then, $CE^*(P)\geq CE^*(Q)$ if $P\trianglerighteq Q$.

Proof Let $U/SIM(P)=\{S_P(u_1),S_P(u_2),\ldots,S_P(u_{|U|})\}$ and $U/SIM(Q)=\{S_Q(u_1),S_Q(u_2),\ldots,S_Q(u_{|U|})\}$. It follows from $P\trianglerighteq Q$ that there exists a sequence $\{S_Q(u'_1),S_Q(u'_2),\ldots,S_Q(u'_{|U|})\}$ such that $|S_P(u_i)|\leq|S_Q(u'_i)|$, $\forall i\in\{1,2,\ldots,|U|\}$. Hence, we have that

$$\begin{aligned}CE^*(P)&=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|U|}^2-C_{|S_P(u_i)|}^2}{C_{|U|}^2}=1-\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|S_P(u_i)|}^2}{C_{|U|}^2}\\&\geq1-\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|S_Q(u'_i)|}^2}{C_{|U|}^2}=1-\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|S_Q(u_i)|}^2}{C_{|U|}^2}\\&=\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{C_{|U|}^2-C_{|S_Q(u_i)|}^2}{C_{|U|}^2}=CE^*(Q)\end{aligned}$$

i.e., $CE^*(P)\geq CE^*(Q)$. This completes the proof. ■

Theorem 6.12 states that the combination entropy CE^* of an incomplete information system increases as the information granularity of this system become finer.

Corollary 6.6 Let $S=(U,A)$ be an incomplete information system and$P$, $Q\subseteq A$. Then, $CE^*(P)<CE^*(Q)$if $G(P)>G(Q)$.

6. Conclusions

According to partial ordering $\preceq$, every class in one approximation space is requested to be contained within a corresponding class in the other approximation space in an information system. But this restriction could not felicitously depict the scale of information granulation of an approximation space in information systems. For this reason, a new partial ordering $\trianglerighteq$ is introduced. We show the mechanism of how this partial ordering overcomes the limitation of $\preceq$ by several illustrative examples. We also point out that the granulation monotonicity induced by $\trianglerighteq$ is all satisfied by all existing information granulations. As a result, a uniform representation of the partial orderings is obtained for characterizing the relationship among approximation spaces, which may lead to a strategy for dealing with information granulations in some real-word applications that involve complete information, incomplete information and maximal consistent blocks all together. With the above discussions, we develop the theoretical foundation of granular computing in information systems for its further research. Another important thing we should point out is that partial ordering $\trianglerighteq$ presented in this paper is the natural generalization of partial ordering $\preceq$.

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