Applied Mathematics and Computation 218 (2012) 5409–5413

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Embedding long cycles in faulty k-ary 2-cubes ☆

Shiying Wang [a,*], Kai Feng [b], Shurong Zhang [a], Jing Li [a]

[a] School of Mathematical Sciences, Shanxi University, Taiyuan, Shanxi 030006, People's Republic of China
[b] School of Computer and Information Technology, Shanxi University, Taiyuan, Shanxi 030006, People's Republic of China

ARTICLE INFO

Keywords:
Fault-tolerance
Cycle embeddings
k-Ary 2-cubes
Torus

ABSTRACT

The class of k-ary n-cubes represents the most commonly used interconnection topology for distributed-memory parallel systems. Given an even $k \geqslant 4$, let (V_1, V_2) be the bipartition of the k-ary 2-cube, f_{v1}, f_{v2} be the numbers of faulty vertices in V_1 and V_2, respectively, and f_e be the number of faulty edges. In this paper, we prove that there exists a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the k-ary 2-cube with $0 \leqslant f_{v1} + f_{v2} + f_e \leqslant 2$. This result is optimal with respect to the number of faults tolerated.

1. Introduction

The k-ary n-cube Q_n^k has many desirable properties, such as ease of implementation, low-latency and high-bandwidth inter-processor communication. Therefore, a number of distributed-memory parallel systems (also known as multicomputers) have been built with a k-ary n-cube forming the underlying topology. The graph embedding is a technique that maps a guest graph into a host graph (usually an interconnection architecture). Many graph embeddings take cycles and paths as guest graphs [1,3–10], because these interconnection architectures are widely used in distributed-memory parallel systems.

As failures are inevitable, fault-tolerance is an important issue in the distributed-memory parallel system. Recently, fault-tolerant cycle-embeddings of various interconnection networks received much attention (see, for example, [1,3,9]). In [11], Yang et al. proved that the faulty k-ary 2-cube with odd $k \geqslant 3$ admits a hamiltonian cycle if the number of faults does not exceed 2. For even $k \geqslant 4$, Stewart and Xiang [7] investigated the problem of embedding long paths in k-ary 2-cubes with faulty vertices and edges and presented the following result.

Theorem 1 [7]. *Let $k \geqslant 4$ be even, and let f_v be the number of faulty vertices and f_e be the number of faulty edges in Q_2^k with $0 \leqslant f_v + f_e \leqslant 2$. Given any two healthy vertices s and t of Q_2^k, then there is a path from s to t of length at least $k^2 - 2f_v - 1$ in the faulty Q_2^k if s and t have different parities (the parity of a vertex $v = v_1v_2$ of Q_2^k is defined to be $v_1 + v_2$ modulo 2).*

As every pair of adjacent vertices have different parities when k is even, there is a cycle of length at least $k^2 - 2f_v$ in the faulty Q_2^k. In fact, this result can be improved according to the possible distribution of the faulty vertices. The parity of a vertex $v = v_1v_2$ of Q_2^k is defined to be $v_1 + v_2$ modulo 2. We speak of a vertex as being odd or even according to whether its parity is odd or even. In this paper, we prove that there exists a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the Q_2^k with at most two faults, where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd). As $f_{v1} + f_{v2} = f_v$, we have $\max\{f_{v1}, f_{v2}\} < f_v$ when $f_{v1} = f_{v2} = 1$. Obviously, $k^2 - 2\max\{f_{v1}, f_{v2}\} \geqslant k^2 - 2f_v$. Therefore, our result improves the result noted above.

☆ This work is supported by the Natural Science Foundation of China (61070229).
* Corresponding author.
E-mail address: shiying@sxu.edu.cn (S. Wang).

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.11.026

The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions. In Section 3, we prove the main result. Conclusions are covered in Section 4.

2. Basic definitions

Throughout this paper, notation and terminology mostly follow [2].

The k-ary 2-cube Q_2^k is a graph consisting of k^2 vertices, each has the form $v = v_1v_2$, where $0 \leqslant v_1, v_2 \leqslant k-1$. Two vertices $v = v_1v_2$ and $u = u_1u_2$ are adjacent if and only if there exists an integer j, $j \in \{1,2\}$, such that $u_j = v_j \pm 1 \pmod{k}$ and $u_i = v_i$, for $i \in \{1,2\} \backslash \{j\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. A k-ary 2-cube with even $k \geqslant 4$ is a bigraph. Let V_1 (resp. V_2) be the set of the vertices which are even (resp. odd). Then (V_1, V_2) is a bipartition of the k-ary 2-cube. Many structural properties of k-ary 2-cubes are known, but of particular relevance for us is that a k-ary 2-cube is vertex-transitive, that is, for any two distinct vertices u and v of Q_2^k, there is an automorphism of Q_2^k mapping u to v. In particular, the mapping $\theta : ij \to i(k-j)$, $0 \leqslant i, j \leqslant k-1$, is an automorphism of Q_2^k.

For convenience, we write $v_{a,b}$ as the vertex of Q_2^k with the form ab, where $0 \leqslant a, b \leqslant k-1$. For $0 \leqslant i \leqslant j \leqslant k-1$, Row$(i:j)$ is the subgraph of Q_2^k induced by $\{v_{a,b} : i \leqslant a \leqslant j, 0 \leqslant b \leqslant k-1\}$. We simply write Row$(i)$ instead of Row$(i:i)$. It can be seen that each Row(i) is a cycle of length k. Let $(v_{i,j}, v_{i,j+1})$ be an edge of Row(i). Then the edge $(v_{m,j}, v_{m,j+1})$, $m \in \{i-1, i+1\}$, is called the corresponding edge of $(v_{i,j}, v_{i,j+1})$ in Row(m).

3. Cycle embeddings in faulty *k*-ary 2-cubes

To show our main result, we first introduce some useful lemmas. A pair of vertices $\{u, v\}$ is odd (resp. even) if u and v have different (resp. the same) parities.

According to the proof of Lemma 1 in [7], the following lemma holds.

Lemma 3.1 [7]. *Given an even $k \geqslant 4$, let v be a faulty vertex of Row(0:1) in Q_2^k and let s and t be two distinct healthy vertices of Row(0:1). If $\{v,s\}$ is odd and $\{s,t\}$ is even, then there is a path from s to t of length $2k-2$ in Row(0:1).*

According to Theorem 1, the following lemma holds.

Lemma 3.2 [7]. *Given an even $k \geqslant 4$, let s and t be any two distinct healthy vertices of Q_2^k with two faulty vertices. Then there is a path of length at least $k^2 - 5$ from s to t if $\{s,t\}$ is odd.*

Lemma 3.3 [7]. *Given an even $k \geqslant 4$, let s and t be two distinct healthy vertices of Row$(0:p-1)$ in Q_2^k, where $2 \leqslant p \leqslant k$. If $\{s,t\}$ is odd (resp. even), then there is a path from s to t of length $pk-1$ (resp. $pk-2$) in Row$(0:p-1)$.*

A matching in a graph is a set of pairwise nonadjacent edges. The vertex incident with an edge of a matching is said to be covered by the matching. A perfect matching is one which covers every vertex of the graph. Let G_1 and G_2 be two graphs. We denote by $G_1 \triangle G_2$ the graph induced by the edges of $E(G_1) \triangle E(G_2)$, where $E(G_1) \triangle E(G_2)$ denotes the symmetric difference of $E(G_1)$ and $E(G_2)$. Given an integer m with $1 \leqslant m \leqslant k$, let $M = \{(v_{i,j_1}, v_{i,j_1+1}), (v_{i,j_2}, v_{i,j_2+1}), \ldots, (v_{i,j_m}, v_{i,j_m+1}) : 0 \leqslant j_l \leqslant k-1, l = 1, 2, \ldots, m\} \subseteq E(\text{Row}(i))$ and let $C_{j_n} = (v_{i,j_n}, v_{i+1,j_n}, v_{i+1,j_n+1}, v_{i,j_n+1}, v_{i,j_n})$. Set $\mathcal{C}(M) = C_{j_1} \triangle C_{j_2} \triangle \ldots \triangle C_{j_m}$.

Lemma 3.4. *Given an even $k \geqslant 4$, let v and w be two distinct faulty vertices of Row(0:1) in Q_2^k. If $\{v,w\}$ is odd, then there is a cycle of length $2k-2$ in Row(0:1) that contains at least one healthy edge of Row(0).*

Proof. Without loss of generality, assume that $v = v_{0,0}$. We distinguish two cases.

Case 1. w is a vertex of Row(0). Let $w = v_{0,i} (1 \leqslant i \leqslant k-1)$. As $\theta : v_{i,j} \to v_{i,k-j}$ is an automorphism of Q_2^k, $\theta(v_{0,0}) = v_{0,0}$ and $\theta(v_{0,i}) = v_{0,k-i}$, it is enough to consider $1 \leqslant i \leqslant \frac{k}{2}$. As $\{v,w\}$ is odd and $v_{0,0}$ is even, i is odd. Let $M = \{(v_{0,1}, v_{0,2}), (v_{0,3}, v_{0,4}), \ldots, (v_{0,i-2}, v_{0,i-1}), (v_{0,i+1}, v_{0,i+2}), \ldots, (v_{0,k-2}, v_{0,k-1})\}$. Then it is easy to see that M is a perfect matching of Row(0) $- \{v,w\}$. Thus, $C = \text{Row}(1) \triangle \mathcal{C}(M)$ is a cycle of length $2k-2$ in Row(0:1). As $|M| = \frac{k-2}{2} \geqslant 1$ and $M \subseteq E(C)$, C contains at least one healthy edge of Row(0).

Case 2. w is a vertex of Row(1). Let $w = v_{1,j} (0 \leqslant j \leqslant k-1)$. As $\{v,w\}$ is odd and $v_{0,0}$ is even, j is even. Suppose that $w = v_{1,0}$, then $C = (v_{0,1}, v_{0,2}, \ldots, v_{0,k-2}, v_{0,k-1}, v_{1,k-1}, v_{1,k-2}, \ldots, v_{1,2}, v_{1,1}, v_{0,1})$ is as required. Suppose that $w \neq v_{1,0}$. Let M_1' be the maximum matching of Row(1) $- w$ such that $(v_{1,0}, v_{1,1}) \in M_1'$ and $(v_{1,k-2}, v_{1,k-1}) \notin M_1'$. Set $M_1 = M_1' \cup \{(v_{1,0}, v_{1,k-1})\}$. Let M_0 be the set of corresponding edges of M_1 in Row(0). Then $C = \text{Row}(1) \triangle \mathcal{C}(E(\text{Row}(0)) - M_0)$ is a cycle of length $2k-2$ in Row(0:1). As $|M_0| = |M_1| < k$ and $(E(\text{Row}(0)) - M_0) \subseteq E(C)$, C contains at least one healthy edge of Row(0). □

Theorem 3.1. *Let $k \geqslant 4$ be even, and let $f_v \leqslant 2$ be the number of faulty vertices in Q_2^k. Then there is a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k, where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd) and $f_{v1} + f_{v2} = f_v$.*

Proof. According to the number of faulty vertices f_v, we distinguish three cases.

Case 1. $f_v = 0$.
In this case, $f_{v1} + f_{v2} = 0$. We have $f_{v1} = f_{v2} = 0$. Note that Row$(0{:}k-1)$ is not empty. Let $(s,t) \in E(\text{Row}(0{:}k-1))$ be a healthy edge in Q_2^k. Then $\{s,t\}$ is odd. By Lemma 3.3, there is a path P in Row$(0{:}k-1)$ from s to t of length k^2-1. So $P+(s,t)$ is a cycle of length k^2 in Q_2^k, where $k^2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$.

Case 2. $f_v = 1$.
In this case, $f_{v1} + f_{v2} = 1$. We have

$$\begin{cases} f_{v1} = 1, \\ f_{v2} = 0, \end{cases} \quad \text{or} \quad \begin{cases} f_{v1} = 0, \\ f_{v2} = 1. \end{cases} \tag{3.1}$$

Without loss of generality, we consider $f_{v1} = 1$ and $f_{v2} = 0$. Let even $v = v_{0,0}$ be the faulty vertex. Note that Row(0:1) and Row$(2{:}k-1)$ are two subgraphs of Q_2^k. Given an odd i and an even j $(0 \leqslant i,j \leqslant k-1)$, let $s = v_{0,i}$ and $t = v_{1,j}$. Then $\{v,s\}$ is odd and $\{s,t\}$ is even. By Lemma 3.1, there is a path P_1 in Row(0:1) from s to t of length $2k-2$. Let $s_1 = v_{k-1,i}$ and $t_1 = v_{2,j}$. Then $(v_{0,i}, v_{k-1,i})$ and $(v_{1,j}, v_{2,j})$ are two healthy edges in the faulty Q_2^k. It is easy to see that $\{s_1,t_1\}$ is even. Combining this with the fact that Row$(2{:}k-1)$ and Row$(0{:}k-3)$ are isomorphic, Lemma 3.3 implies that there is a path P_2 in Row$(2{:}k-1)$ from s_1 to t_1 of length $(k-2)k-2$. Now, $P_1 \cup P_2 + \{(v_{0,i}, v_{k-1,i}), (v_{1,j}, v_{2,j})\}$ is a cycle of length l with $l = (2k-2) + (k-2)k - 2 + 2 = k^2 - 2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k.

Case 3. $f_v = 2$. In this case, $f_{v1} + f_{v2} = 2$. We have

$$\begin{cases} f_{v1} = 2, \\ f_{v2} = 0, \end{cases} \quad \text{or} \quad \begin{cases} f_{v1} = 1, \\ f_{v2} = 1, \end{cases} \quad \text{or} \quad \begin{cases} f_{v1} = 0, \\ f_{v2} = 2. \end{cases} \tag{3.2}$$

Case 3.1. $f_{v1} = 2, f_{v2} = 0$ or $f_{v1} = 0, f_{v2} = 2$. Without loss of generality, we consider $f_{v1} = 2 f_{v2} = 0$. Let $(s,t) \in E(Q_2^k)$ be a healthy edge. Then $\{s,t\}$ is odd. By Lemma 3.2, there is a path P in the faulty Q_2^k from s to t of length k^2-5. So $P+(s,t)$ is a cycle of length k^2-4 in the faulty Q_2^k, where $k^2-4 = k^2 - 2\max\{f_{v1}, f_{v2}\}$.

Case 3.2. $f_{v1} = f_{v2} = 1$.
Suppose that even v and odd w are the two faulty vertices. Without loss of generality, let $v = v_{0,0}$. According to the possible distribution of w, we distinguish three subcases.

Case 3.2.1. w is a vertex of Row(0:1) or Row$(k-1)$.
Suppose that w is a vertex of Row(0:1). Note that Row(0:1) and Row$(2{:}k-1)$ are two subgraphs of Q_2^k. As $\{v,w\}$ is odd, Lemma 3.4 implies that there is a cycle C_1 in Row(0:1) of length $2k-2$ containing a healthy edge (v',w') of Row(0). Let v'_1 (resp. w'_1) be the adjacent vertex of v' (resp. w') in Row$(k-1)$. Then (v', v'_1) and (w', w'_1) are two healthy edges in the faulty Q_2^k. As $\{v',w'\}$ is odd, we have that $\{v'_1, w'_1\}$ is odd. Combining this with the fact that Row$(2{:}k-1)$ and Row$(0{:}k-3)$ are isomorphic, Lemma 3.3 implies that there is a path P in Row$(2{:}k-1)$ from v'_1 to w'_1 of length $(k-2)k-1$. So $C_1 \cup P + \{(v', v'_1), (w', w'_1)\} - \{(v', w')\}$ is a cycle of length l with $l = (2k-2) + (k-2)k - 1 + 2 - 1 = k^2 - 2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k.
Suppose that $w \in V(\text{Row}(k-1))$. Let $w = v_{k-1,j}$ $(0 \leqslant j \leqslant k-1)$. As $\theta_1 : v_{i,j} \to v_{k-i,j}$ is an automorphism of Q_2^k, $\theta_1(v_{0,0}) = v_{0,0}$ and $\theta_1(v_{k-1,j}) = v_{1,j}$. Similarly, we can obtain a desired cycle.

Case 3.2.2. w is a vertex of Row(2:3) or Row$(k-3{:}k-2)$. Suppose that w is a vertex of Row(2:3). For $k=4$, note that Row(0:1) and Row(2:3) are subgraphs of Q_2^4. Let even w' be a healthy vertex of Row(2) and let w'_1 be the adjacent vertex of w' in Row(1). Then w'_1 is odd and (w', w'_1) is a healthy edge in the faulty Q_2^4. Let odd v' be a healthy vertex of Row(0). As $\{v, v'\}$ is odd and $\{v', w'_1\}$ is even, Lemma 3.1 implies that there is a path P_1 in Row(0:1) from v' to w'_1 of length 6. Let v'_1 be the adjacent vertex of v' in Row(3). Then v'_1 is even and (v', v'_1) is a healthy edge in the faulty Q_2^4. As $\{w,w'\}$ is odd and $\{w', v'_1\}$ is even, Lemma 3.1 implies that there is a path P_2 in Row(2:3) from v'_1 to w' of length 6. So $P_1 \cup P_2 + \{(w', w'_1), (v', v'_1)\}$ is a cycle of length l with $l = 6+6+2 = 14 = 4^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^4.
For even $k \geqslant 6$, note that Row(0:1), Row(2:3) and Row$(4{:}k-1)$ are subgraphs of Q_2^k. Let even w' be a healthy vertex of Row(2) and let w'_1 be the adjacent vertex of w' in Row(1). Then w'_1 is odd and (w', w'_1) is a healthy edge in the faulty Q_2^k. Let odd v' be a healthy vertex of Row(0). As $\{v, v'\}$ is odd and $\{v', w'_1\}$ is even, Lemma 3.1 implies that there is a path P_1 in Row(0:1) from v' to w'_1 of length $2k-2$. Let even x be a healthy vertex of Row(3). As $\{w,w'\}$ is odd and $\{w',x\}$ is even, Lemma 3.1 implies that there is a path P_2 in Row(2:3) from x to w' of length $2k-2$. Let x_1 be the adjacent vertex of x in Row(4). Then x_1 is odd and (x,x_1) is a healthy edge in the faulty Q_2^k. Let v'_1 be the adjacent vertex of v' in Row$(k-1)$. Then v'_1 is even and (v', v'_1) is a healthy edge in the faulty Q_2^k. As $\{x_1, v'_1\}$ is odd and Row$(4{:}k-1)$ and Row$(0{:}k-5)$ are isomorphic, Lemma 3.3 implies that there is a path P_3 in Row$(4{:}k-1)$ from x_1 to v'_1 of length $(k-4)k-1$. So $P_1 \cup P_2 \cup P_3 + \{(w', w'_1), (x, x_1), (v', v'_1)\}$ is a cycle of length l with

$l = (2k-2) + (2k-2) + (k-4)k - 1 + 3 = k^2 - 2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k.

Suppose that $w \in V(\mathrm{Row}(k-3{:}k-2))$. Let $w = v_{i,j}$, where $i \in \{k-3, k-2\}$ and $0 \leqslant j \leqslant k-1$. As $\theta_1: v_{i,j} \rightarrow v_{k-i,j}$ is an automorphism of Q_2^k, $\theta_1(v_{0,0}) = v_{0,0}$, $\theta_1(v_{k-3,j}) = v_{3,j}$ and $\theta_1(v_{k-2,j}) = v_{2,j}$. Similarly, we can obtain a desired cycle.

Case 3.2.3. w is a vertex of Row(n), where $4 \leqslant n \leqslant k-4$. In this case, we have even $k \geqslant 8$. Note that Row(0:1), Row($2{:}n-1$), Row($n{:}n+1$) and Row($n+2{:}k-1$) are subgraphs of Q_2^k. Let odd v' (resp. odd w') be a healthy vertex of Row(0) (resp. Row(1)). As $\{v, v'\}$ is odd and $\{v', w'\}$ is even, Lemma 3.1 implies that there is a path P_1 in Row(0:1) from v' to w' of length $2k-2$. Let even v_1' (resp. even w_1') be a healthy vertex of Row(n) (resp. Row($n+1$)). As $\{w, v_1'\}$ is odd and $\{v_1', w_1'\}$ is even, Lemma 3.1 implies that there is a path P_2 in Row($n{:}n+1$) from v_1' to w_1' of length $2k-2$. Let w'' (resp. v_1'') be the adjacent vertex of w' (resp. v_1') in Row(2) (resp. Row($n-1$)). Then (w', w'') and (v_1', v_1'') are two healthy edges in the faulty Q_2^k. It is easy to see that w'' is even and v_1'' is odd. So $\{w'', v_1''\}$ is odd. Combining this with the fact that Row($2{:}n-1$) and Row($0{:}n-3$) are isomorphic, Lemma 3.3 implies that there is a path P_3 in Row($2{:}n-1$) from w'' to v_1'' of length $(n-2)k-1$. Let v'' (resp. w_1'') be the adjacent vertex of v' (resp. w_1') in Row($k-1$) (resp. Row($n+2$)). Then (w_1', w_1'') and (v', v'') are two healthy edges in the faulty Q_2^k. It is easy to see that v'' is even and w_1'' is odd. So $\{v'', w_1''\}$ is odd. Combining this with the fact that Row($n+2{:}k-1$) and Row($0{:}k-n-3$) are isomorphic, Lemma 3.3 implies that there is a path P_4 in Row($n+2{:}k-1$) from v'' to w_1'' of length $(k-n-2)k-1$. So $P_1 \cup P_2 \cup P_3 \cup P_4 + \{(w', w''), (v_1', v_1''), (w_1', w_1''), (v', v'')\}$ is a cycle of length l with $l = (2k-2) + (2k-2) + (n-2)k - 1 + (k-n-2)k - 1 + 4 = k^2 - 2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k. As we have discussed all possible cases, the proof is complete. □

According to Theorem 1, the following three lemmas hold.

Lemma 3.5 [7]. *Given an even $k \geqslant 4$, let s and t be any two distinct healthy vertices of Q_2^k with one faulty edge. Then there is a path in the faulty Q_2^k from s to t of length k^2-1 if $\{s, t\}$ is odd.*

Lemma 3.6 [7]. *Given an even $k \geqslant 4$, let s and t be any two distinct healthy vertices of Q_2^k with two faulty edges. Then there is a path in the faulty Q_2^k from s to t of length k^2-1 if $\{s, t\}$ is odd.*

Lemma 3.7 [7]. *Given an even $k \geqslant 4$, let s and t be any two distinct healthy vertices of Q_2^k with one faulty edge and one faulty vertex. Then there is a path in the faulty Q_2^k from s to t of length at least k^2-3 if $\{s, t\}$ is odd.*

Note that the Q_2^k with at most two faults is not empty and every pair of adjacent vertices have different parities when k is even. When $f_v = 0$ and $f_e = 1$ or $f_e = 2$, it follows from Lemmas 3.5 and 3.6 that each healthy edge of the faulty Q_2^k lies on a cycle of length $k^2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$. When $f_v = 1$ and $f_e = 1$, it follows from Lemma 3.7 that each healthy edge of the faulty Q_2^k lies on a cycle of length $k^2 - 2 = k^2 - 2\max\{f_{v1}, f_{v2}\}$. Combining these with Theorem 3.1, it can be seen that the following theorem holds.

Theorem 3.2. *Let $k \geqslant 4$ be even, and let f_v be the number of faulty vertices and f_e be the number of faulty edges in Q_2^k with $0 \leqslant f_v + f_e \leqslant 2$. Then there is a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty Q_2^k, where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd) and $f_{v1} + f_{v2} = f_v$.*

This result is optimal. The reasons are as follows. (1) Suppose that Q_2^k contains no faulty vertices or two faulty vertices with different parities. Then the cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ is a hamiltonian cycle of the faulty Q_2^k. (2) Suppose that Q_2^k contains exactly one faulty vertex. Recall that a Q_2^k with even $k \geqslant 4$ is actually a bigraph. As there is no cycle of odd length, the length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ of the cycle cannot be improved. (3) Suppose that Q_2^k contains two faulty vertices with the same

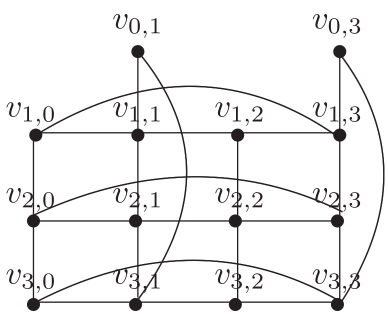

Fig. 1. The Q_2^4 with two faulty vertices $v_{0,0}$ and $v_{0,2}$.

parities. Similarly, there is no cycle of odd length in the faulty Q_2^k. In addition, it is not difficult to see that the Q_2^4 with two faulty vertices $v_{0,0}$ and $v_{0,2}$ has no cycle of length $k^2 - 2 = 14$ (See Fig. 1). Therefore, the length $k^2 - 4 = k^2 - 2\max\{f_{v1}, f_{v2}\}$ of the cycle cannot be improved.

4. Conclusions

In this paper, we investigate the problem of embedding long cycles in k-ary 2-cubes with f_v faulty vertices and f_e faulty edges, where $k \geqslant 4$ is even and $0 \leqslant f_v + f_e \leqslant 2$. It is proved that there is a cycle of length $k^2 - 2\max\{f_{v1}, f_{v2}\}$ in the faulty k-ary 2-cube where f_{v1} (resp. f_{v2}) is the number of faulty vertices which are even (resp. odd). Our result improves the result in the case when $n = 2$ given by Stewart and Xiang [7]. A generalization would lead to looking for the long cycles in the faulty k-ary n-cubes with $n \geqslant 3$ and even $k \geqslant 4$.

Acknowledgement

The authors would like to express their deepest gratitude to the anonymous referees for the constructive suggestions and comments that improve the quality of this paper.

References

[1] Yaagoub A. Ashir, Iain A. Stewart, Fault-tolerant embeddings of hamiltonian circuits in *k*-ary *n*-cubes, SIAM Journal on Discrete Mathematics 15 (3) (2002) 317–328.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2007.
[3] Jou-Ming Chang, Jinn-Shyong Yang, Fault-tolerant cycle-embedding in alternating group graphs, Applied Mathematics and Computation 197 (2008) 760–767.
[4] Jywe-Fei Fang, The bipancycle-connectivity of the hypercube, Information Sciences 178 (24) (2008) 4679–4687.
[5] Sun-Yuan Hsieh, Jung-Yiau Shiu, Cycle embedding of augmented cubes, Applied Mathematics and Computation 191 (2007) 314–319.
[6] Meijie Ma, Guizhen Liu, Xiangfeng Pan, Path embedding in faulty hypercubes, Applied Mathematics and Computation 192 (2007) 233–238.
[7] Iain A. Stewart, Yonghong Xiang, Embedding long paths in *k*-ary *n*-cubes with faulty nodes and links, IEEE Transactions on Parallel and Distributed Systems 19 (8) (2008) 1071–1085.
[8] Iain A. Stewart, Y.H. Xiang, Bipanconnectivity and Bipancyclicity in *k*-ary *n*-cubes, IEEE Transactions on Parallel and Distributed Systems 20 (1) (2009) 25–33.
[9] Chang-Hsiung. Tsai, Cycles embedding in hypercubes with node failures, Information Processing Letters 102 (6) (2007) 242–246.
[10] Shiying Wang, Shangwei Lin, Path embeddings in faulty 3-ary *n*-cubes, Information Sciences 180 (1) (2010) 191–197.
[11] Ming-Chien Yang, Jimmy J.M. Tan, Lih-Hsing Hsu, Hamiltonian circuit and linear array embeddings in faulty *k*-ary *n*-cubes, Journal of Parallel and Distributed Computing 67 (4) (2007) 362–368.