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Global stability of periodic solutions for a discrete predator–prey system with functional response

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Abstract The purpose of this paper is to study the existence and global stability of a periodic solution for a discrete predator–prey system with the Beddington–DeAngelis functional response and predator cannibalism. By using the continuation theorem, the existence conditions of at least one periodic solution are obtained, and the sufficient conditions, which ensure the global stability of the positive periodic solution, are derived by constructing a special Lyapunov function.

Keywords Predator–prey · Predator cannibalism · Periodic solution · Global stability

1 Introduction

The dynamic relationship between predators and their prey has long been and will continue to be the dominant themes in both ecology and mathematical biology. Since the great work of Lotka (in 1925) and Volterra (in 1926), modeling these interactions has been one of the central themes in mathematical biology [1, 2].

Among predator–prey relationships, one significant component is the functional response of predators. In general, the functional response can be either prey dependent or predator dependent. Holling family ones are prey dependent, which may fail to model the interference among predators, and have been facing challenges from the biology and physiology communities. The predator-dependent functional responses can provide better descriptions of a predator feeding over a range of predator–prey abundances, as is supported by much significant laboratory and field evidence (see [3] and references therein). However, the Beddington–DeAngelis functional response, first proposed by Beddington [4] and DeAngelis [5], performed even better. As a result, predator–prey systems with the Beddington–DeAngelis response have been studied extensively in the literature [6–10]. The system can be written as

$$\begin{cases} \dfrac{dz(t)}{dt} = rz(t)\big[1 - z(t)\big] - \dfrac{bz(t)p(t)}{Az(t) + Bp(t) + C}, \\ \dfrac{dp(t)}{dt} = p(t)\left[\dfrac{kbz(t)}{Az(t) + Bp(t) + C} - D\right], \end{cases} \tag{1}$$

where r, b, k, A, B, C, D are positive constants and $z(t)$, $p(t)$ represent the population density of prey and predator at time t, respectively.

Recently, some work showed that cannibalism can stabilize population cycles in a Lotka–Volterra type

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predator–prey model [11, 12]. Specifically, Kohlmeier et al. used a Lotka–Volterra type predator–prey model with logistic prey growth and a Beddington–DeAngelis functional response and added a cannibalism term [11]. The system has the following form:

$$\begin{cases} \dfrac{dz(t)}{dt} = rz(t)\big[1-z(t)\big] - \dfrac{bz(t)p(t)}{Az(t)+Bp(t)+C}, \\ \dfrac{dp(t)}{dt} = p(t)\Big[\dfrac{k(t)b(t)z(t)}{A(t)z(t)+B(t)p(t)+C(t)} \\ \qquad - D(t)\Big] - y_1\big(z(t),p(t)\big), \end{cases} \tag{2}$$

where

$$y_1\big(z(t),p(t)\big) = \frac{\theta(t)p^2(t)}{A(t)z(t)+B(t)p(t)+C(t)}, \tag{3}$$

represents predator cannibalism. $z(t)$ and $p(t)$ are the density of prey and predator, respectively; r is the growth rate of the prey population; b and θ are the attack rates on prey and conspecifics; A and B are the handling times of prey and conspecifics; k is the food to newborns conversion factor, and D is the predator death rate. And all the $r(t), b(t), k(t), \theta(t), A(t), B(t), C(t), D(t)$ are positive T-periodic sequences.

Though much progress has been seen for the system (2), there is no work that has been done for such discrete model. However, it is well known that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. In addition, discrete time models can also provide efficient computational models of continuation for numerical simulations [13]. He and Lai investigated the bifurcation and chaotic behavior of a discrete-time predator–prey system [14]. Fang and Chen obtained the permanence of a discrete Lotka–Volterra predator–prey system with delays [15]. Xia et al. showed that the discrete-time analogues preserve the periodicity of the continuous-time models with monotonic functional responses [16]. Ghaziani et al. presented resonance and bifurcation in a discrete-time predator–prey system [17]. Motivated by the previous work, we will analyze the dynamics of a discrete analogue of the continuous system (2) in this paper.

This paper is organized as follows. In Sect. 2, we give the discrete form of system (2). In Sects. 3 and 4, by using the continuation theorem and constructing a Lyapunov function, we obtain the existence conditions and global stability of the positive periodic solution. In Sect. 5, we present a conclusion and some discussion.

2 Discrete analogue of the model

Let us assume that the average growth rates in system (2) change only at regular intervals of time. Then we can incorporate this aspect in system (2) and obtain the following modified system:

$$\begin{cases} \dfrac{1}{z(t)}\dfrac{dz(t)}{dt} = r[t]\big(1-z[t]\big) \\ \qquad - \dfrac{b[t]p[t]}{A[t]z[t]+B[t]p[t]+C[t]}, \\ \dfrac{1}{p(t)}\dfrac{dp(t)}{dt} = \Big(\dfrac{k[t]b[t]z[t]}{A[t]z[t]+B[t]p[t]+C[t]} - D[t]\Big) \\ \qquad - g\big(z[t],p[t]\big), \end{cases} \tag{4}$$

where

$$g(z,p) = \frac{\theta p}{Az+Bp+C}, \tag{5}$$

and $[t]$ denotes the integer part of $t \in (0,+\infty)$. The system (4) is known as a differential equation with piecewise constant arguments, and it occupies a position midway between differential equations and a difference equation.

Thus, on any interval of the form $[n, n+1), n \in N$, we can integrate the system (4) and obtain that for $n \le t < n+1$,

$$\begin{cases} z(t) = z(n)\exp\Big\{r(n)\big[1-z(n)\big] \\ \qquad - \dfrac{b(n)p(n)}{A(n)z(n)+B(n)p(n)+C(n)}\Big\}, \\ p(t) = p(n) \\ \qquad \times \exp\Big\{k(n)\dfrac{b(n)z(n)}{A(n)z(n)+B(n)p(n)+C(n)} \\ \qquad - D(n) - g\big(z(n),p(n)\big)\Big\}. \end{cases} \tag{6}$$

Let $t \to n+1$, and we obtain from (6) that

$$\begin{cases} z(n+1) = z(n)\exp\Big\{r(n)\big[1-z(n)\big] \\ \qquad\qquad - \dfrac{b(n)p(n)}{A(n)z(n)+B(n)p(n)+C(n)}\Big\}, \\ p(n+1) = p(n)\exp\Big\{k(n) \\ \qquad\qquad \times \dfrac{b(n)z(n)}{A(n)z(n)+B(n)p(n)+C(n)} \\ \qquad\qquad - D(n) - g\big(z(n),p(n)\big)\Big\}. \end{cases} \tag{7}$$

Apparently, the system (7) is a discrete analogue of system (2). Throughout the rest of this paper, we assume that $z(0) > 0$, $p(0) > 0$. It is easy to check that the solution $(z(n), p(n))^T$ of the system (7) is positive for all $n \in N$.

In the following sections, we will pay our attention on system (7).

3 Existence of positive periodic solutions

In this section, we will establish the existence condition of at least one T-periodic solution of system (7) by using continuation theorem which was proposed by Gaines and Mawhin [18]. For the sake of discussion, let

$$z(n) = \exp\{x_1(n)\}, \qquad p(n) = \exp\{x_2(n)\},$$

then we can get

$$\begin{cases} x_1(n+1) - x_1(n) = r(n)\big[1-\exp x_1(n)\big] \\ \qquad\qquad - \dfrac{b(n)\exp\{x_2(n)\}}{f(x_1(n),x_2(n))}, \\ x_2(n+1) - x_2(n) = k(n)\dfrac{b(n)\exp\{x_1(n)\}}{f(x_1(n),x_2(n))} \\ \qquad\qquad - D(n) - g_1\big(x_1(n),x_2(n)\big), \end{cases} \tag{8}$$

where

$$f\big(x_1(n),x_2(n)\big) = A(n)\exp\{x_1(n)\} + B(n)\exp\{x_2(n)\} + C(n),$$

$$g_1\big(x_1(n),x_2(n)\big) = \frac{\theta(n)\exp\{x_2(n)\}}{A(n)\exp\{x_1(n)\} + B(n)\exp\{x_2(n)\} + C(n)}.$$

In what follows, we will introduce some symbols and lemmas. Denote N, R, R^2 as the set of natural numbers, real numbers, and a two-dimensional Euclidean vector space, respectively, then we can define the norm of vector $x \in R^2$

$$\|x\| = \sum_{i=1}^{2} x_i,$$

and let

$$I_T = \{0,1,2,\dots,T-1\},$$

$$\overline{w} = \frac{1}{T}\sum_{s=0}^{T-1} w(s), \qquad \overline{W} = \frac{1}{T}\sum_{s=0}^{T-1}\big|w(s)\big|,$$

$$w^u = \max_{s\in I_T} w(s), \qquad w^l = \min_{s\in I_T} w(s),$$

where $w(s)$ is a T-periodic sequence of nonnegative real numbers defined for $s \in N$.

Let the X and Y be two real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm mapping of index zero, and $P : X \to X$, $Q : Y \to Y$, be continuous projections such that $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$, and $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. Denote by L_p the restriction of L to $\operatorname{dom} L \cap \ker P$, $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ the inverse of L_p, and by $J : \operatorname{Im} Q \to \ker L$ an isomorphism of $\operatorname{Im} Q$ onto $\ker L$.

Lemma 1 [18] *Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ a continuous operator, which is L-compact on $\overline{\Omega}$ (i.e., $QN : \overline{\Omega} \to Y$ and $K_p(I-Q)N : \overline{\Omega} \to X$ are compact). Assume*

(a) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in (\operatorname{dom} L \setminus \ker L \cap \partial\Omega) \times (0,1)$;*
(b) *$QNx \neq 0$ for every $x \in \ker L \cap \partial\Omega$;*
(c) *$\deg(QN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Lemma 2 [19] *Let $w : Z \to R$ be T-periodic, i.e., $w(k+T) = w(k)$. Then for any fixed $k_1, k_2 \in I_T$, and any $k \in Z$, one has*

$$w(k) \le w(k_1) + \sum_{s=0}^{T-1}\big|w(s+1)-w(s)\big|,$$

$$w(k) \ge w(k_2) - \sum_{s=0}^{T-1}\big|w(s+1)-w(s)\big|.$$

Now we can define L to be the linear operator from $\operatorname{dom} L \subset X$ to Y with

$$\operatorname{dom} L = \{x = x(n) : x(n) \in R^2, n \in N\}$$

and

$$L(x) = x(n+1) - x(n), \quad x = (x_1, x_2)^T \in \operatorname{dom} L.$$

Define the nonlinear operator $N : X \to Y$ by

$$N(x) = N\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} = \begin{pmatrix} r(n)(1 - \exp\{x_1(n)\}) - \frac{b(n)\exp\{x_2(n)\}}{f(x_1(n),x_2(n))} \\ k(n)\frac{b(n)\exp\{x_1(n)\}}{f(x_1(n),x_2(n))} - D(n) - g_1(x_1(n), x_2(n)) \end{pmatrix}.$$

Then we can consider the operator equation

$$L(x) = \lambda N(x), \quad \lambda \in (0,1). \tag{9}$$

It is trivial to see that L is a bounded linear operator with

$$\ker L = \{x \in \operatorname{dom} L : x(n) = l \in R^2, \ n \in N\},$$

$$\operatorname{Im} L = \left\{x \in \operatorname{dom} L : \sum_{n=0}^{T-1} u(n) = 0, \ n \in N\right\},$$

and

$$\dim \operatorname{Ker} L = 2 = co \dim \operatorname{Im} L.$$

As a result, it follows that L is a Fredholm mapping of index zero.

Define $P : X \to X$, $Q : Y \to Y$ respectively as

$$Px = \frac{1}{T}\sum_{s=0}^{T-1} x(s), \quad x \in X,$$

and

$$Qy = \frac{1}{T}\sum_{s=0}^{T-1} y(s), \quad y \in Y.$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \ker L, \qquad \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q).$$

It follows that $L_{\operatorname{dom} L \cap \ker P} : (I - P)X \to \operatorname{Im} L$ is invertible. And we denote the generalized inverse (to L) by $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ and K_P has the following form:

$$K_p(y) = \sum_{s=0}^{T-1} y(s) - \frac{1}{T}\sum_{s=0}^{T-1}(T - s)y(s).$$

It is obvious that QN and $K_P(I-P)N$ are continuous. Since X is a finite-dimensional Banach space, by using the Arzela–Ascoli theorem, we can easily obtain that $\overline{K_P(I-P)N(\overline{\Omega})}$ is compact for any open bounded $\Omega \subset X$. In addition, $QN(\overline{\Omega})$ is clearly bounded, which implies that N is L-compact on $\overline{\Omega}$.

For the application of Lemma 1, we must search for an appropriate open, bounded subset Ω.

Suppose that $x(n) = (x_1(n), x_2(n))^T \in X$ is an arbitrary solution of system (8) for certain $\lambda \in (0,1)$. Summing both sides of system (8) over the interval $[0, T-1]$ with respect to n, we obtain

$$\overline{r}T = \sum_{n=0}^{T-1}\left[r(n)\exp\{x_1(n)\} + \frac{b(n)\exp\{x_2(n)\}}{f(x_1(n), x_2(n))}\right], \tag{10}$$

$$\overline{D}T = \sum_{n=0}^{T-1}\left[k(n)\frac{b(n)\exp\{x_1(n)\}}{f(x_1(n), x_2(n))} - g_1\big(x_1(n), x_2(n)\big)\right]. \tag{11}$$

From (8) and (10), it follows that

$$\begin{aligned}\sum_{n=0}^{T-1}\left|x_1(n+1) - x_1(n)\right| &\le \left\{\sum_{n=0}^{T-1}\left|r(n)\right| + \sum_{n=0}^{T-1}\left[r(n)\exp\{x_1(n)\} + \frac{b(n)\exp\{x_2(n)\}}{f(x_1(n), x_2(n))}\right]\right\} \\ &= (\overline{R} + \overline{r})T.\end{aligned} \tag{12}$$

Similarly, from (8) and (11), we have

$$\sum_{n=0}^{T-1}\left|x_2(n+1) - x_2(n)\right| \le (\overline{B} + \overline{D})T. \tag{13}$$

In view of $x=\{x(n)\}\in X$, there exist $\xi_i,\eta_i\in I_T$ such that

$$x_i(\xi_i)=\min_{n\in I_T}\{x_i(n)\},\qquad x_i(\eta_i)=\max_{n\in I_T}\{x_i(n)\},\quad i=1,2. \tag{14}$$

From (10) and (14), we can obtain that

$$\bar{r}T\ge\sum_{n=0}^{T-1}\big[r(n)\exp\{x_1(\xi_1)\}\big]=T\overline{(r/L)}\exp\{x_1(\xi_1)\},$$

so then

$$x_1(\xi_1)\le\ln\big[\bar{r}/\overline{(r/L)}\big]:=Q_1. \tag{15}$$

By virtue of (12) and (15) and Lemma 2, we have

$$\begin{aligned}x_1(n)&\le x_1(\xi_1)+\sum_{n=0}^{T-1}\big|x_1(n+1)-x_1(n)\big|\\&\le\ln\big[\bar{r}/\overline{(r/L)}\big]+(\bar{R}+\bar{r})T:=H_1.\end{aligned} \tag{16}$$

On the other hand, from (10) and (14), we can also obtain that

$$\begin{aligned}\bar{r}T&\le\sum_{n=0}^{T-1}\left[r(n)\exp\{x_1(\eta_1)\}+\frac{b(n)}{B(n)}\right]\\&=T\big[\overline{(r/L)}\exp\{x_1(\eta_1)\}+\overline{(b/B)}\big],\end{aligned} \tag{17}$$

which leads to

$$x_1(\eta_1)\ge\ln\big[\big(\bar{r}-\overline{(b/B)}\big)/\overline{(r/L)}\big]:=q_1. \tag{18}$$

This, together with (12) and Lemma 2, yields

$$\begin{aligned}x_1(n)&\ge x_1(\eta_1)-\sum_{n=0}^{T-1}\big|x_1(n+1)-x_1(n)\big|\\&\ge\ln\big[\big(\bar{r}-\overline{(b/B)}\big)/\overline{(r/L)}\big]-(\bar{R}+\bar{r})T:=H_2.\end{aligned} \tag{19}$$

It follows from (16) and (19) that

$$\max_{n\in I_T}\big\{|x_1(n)|\big\}<\max\big\{|H_1|,|H_2|\big\}:=D_1. \tag{20}$$

From (11) and (14), we can obtain

$$\begin{aligned}\bar{D}T&\le\sum_{n=0}^{T-1}\left[k(n)\frac{b(n)\exp\{x_1(n)\}}{f(x_1(n),x_2(n))}\right]\\&\le\sum_{n=0}^{T-1}\left[k(n)\frac{b(n)\exp\{H_1\}}{A^l\exp\{H_2\}+B^l\exp\{x_2(\xi_2)\}}\right]\\&\le\sum_{n=0}^{T-1}\left[\frac{k(n)b(n)\exp\{H_1\}}{B^l\exp\{x_2(\xi_2)\}}\right]\\&=\frac{\exp\{H_1\}}{\exp\{x_2(\xi_2)\}}\overline{(kb/B^l)}T,\end{aligned} \tag{21}$$

which leads to

$$x_2(\xi_2)\le\ln\frac{\overline{(kb/B^l)}}{\bar{D}}+H_1:=Q_2. \tag{22}$$

Together with (13) and Lemma 2, it yields

$$\begin{aligned}x_2(n)&\le x_2(\xi_2)+\sum_{n=0}^{T-1}\big|x_2(n+1)-x_2(n)\big|\\&\le\ln\frac{\overline{(kb/B^l)}}{\bar{D}}+H_1+(\bar{B}+\bar{D})T:=H_3.\end{aligned} \tag{23}$$

We can also obtain from (11) and (14) that

$$\begin{aligned}\bar{D}T&\ge\sum_{n=0}^{T-1}\left[k(n)\frac{b(n)\exp\{H_2\}}{f(x_1(n),x_2(n))}\right]-\overline{(1/H_2)}T\\&\ge\sum_{n=0}^{T-1}\left[k(n)\frac{b(n)\exp\{H_2\}}{1+A^u\exp\{H_1\}+B^u\exp\{x_2(\eta_2)\}}\right]\\&\quad-\overline{(1/H_2)}T,\end{aligned} \tag{24}$$

which leads to

$$\begin{aligned}x_2(\eta_2)\ &\ge\ \ln\left[\frac{\overline{(kb)}\exp\{H_2\}}{B^u(\bar{D}+\overline{(1/H_2)})}-\frac{1+A^u\exp\{H_1\}}{B^u}\right]\\&:=q_2.\end{aligned} \tag{25}$$

This together with (13) and Lemma 2 yields

$$\begin{aligned}x_2(n)&\ge x_2(\eta_2)-\sum_{n=0}^{T-1}\big|x_2(n+1)-x_2(n)\big|\\&\ge q_2-(\bar{B}+\bar{D})T:=H_4.\end{aligned} \tag{26}$$

It follows from (23) and (26) that

$$\max_{n\in I_T}\big\{|x_2(n)|\big\}<\max\big\{|H_3|,|H_4|\big\}:=D_2. \tag{27}$$

Clearly, D_1 and D_2 are independent of λ. Take $D'=D_1+D_2+D_0$ where D_0 is taken sufficiently large that $D_0>|q_1|+|Q_1|+|q_2|+|Q_2|$.

Let $\Omega = \{(x_1(n), x_2(n))^T \in X : \|(x_1, x_2)\| < D'\}$, then Ω is an open, bounded set in X and satisfies the requirement (a) of Lemma 1.

Now, let us consider the algebraic equations

$$\begin{pmatrix} \bar{r} - \overline{(r/L)}\exp\{x_1\} - \mu\frac{1}{T}\sum_{n=0}^{T-1}\frac{b(n)\exp\{x_2\}}{f(x_1,x_2)} \\ -\overline{D} + \frac{1}{T}\sum_{n=0}^{T-1}[\frac{k(n)b(n)\exp\{x_1\}}{f(x_1,x_2)} - g_1(x_1,x_2)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{28}$$

where $\mu \in [0, 1]$ is a parameter and $(x_1, x_2)^T \in R^2$. Similarly, it is easy to show that any solution $(x_1^*, x_2^*)^T$ of (28) with $\mu \in [0, 1]$ satisfies

$$H_2 \le x_1^* \le H_1, \qquad H_4 \le x_2^* \le H_3.$$

Suppose $(x_1, x_2)^T \in \ker L \cap \partial\Omega = R^2 \cap \partial\Omega$, $(x_1, x_2)^T$ is a constant vector in R^2, and

$$\|(x_1, x_2)\| = |x_1| + |x_2| = D',$$

then $QN[(x_1, x_2)^T] \neq 0$, which verifies the requirement (b) of Lemma 1.

Considering the homotopy for computing the Brouwer degree

$$H_\mu[(x_1, x_2)^T] = \mu QN[(x_1, x_2)^T] + (1-\mu)A[(x_1, x_2)^T], \quad \mu \in [0, 1],$$

where

$$A[(x_1, x_2)^T] = \begin{pmatrix} \bar{r} - \overline{(r/L)}\exp\{x_1\} \\ -\overline{D} + \frac{1}{T}\sum_{n=0}^{T-1}[\frac{k(n)b(n)\exp\{x_1\}}{f(x_1,x_2)} - g_1(x_1,x_2)] \end{pmatrix}.$$

By the invariance property of homotopy, direct calculation produces

$$\begin{aligned} \deg(JQN, \ker L \cap \Omega, 0)^+ &= \deg(QN, \ker L \cap \Omega, 0) \\ &= \deg(A, \ker L \cap \Omega, 0) \\ &\neq 0. \end{aligned}$$

We have proved that Ω satisfies all requirement of Lemma 1, then $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, that is, the system (8) has at least one T-periodic solution in $\operatorname{dom} L \cap \overline{\Omega}$, say $(x_1^*(n), x_2^*(n))^T$. Let $z^*(n) = \exp\{x_1^*(n)\}$, $p^*(n) = \exp\{x_2^*(n)\}$, so $(z^*(n), p^*(n))^T$ is an T-periodic solution of system (7) with strictly positive components.

Then we can get the main theorem.

Theorem 1 *If $\bar{r} - \overline{(b/B)} \ge 0$ and $\frac{\overline{(kb)}\exp\{H_2\}}{B^u(\overline{D}+\overline{(1/H_2)})} - \frac{1+A^u\exp\{H_1\}}{B^u} \ge 0$, then the system (7) has at least one positive T-periodic solution.*

4 Global stability of the positive periodic solution

In this section, we obtain the sufficient conditions which guarantee that the positive periodic solution of (7) is globally stable by constructing a suitable Lyapunov function.

From the above discussion, it is easy to get

$$\begin{aligned} H_2' &= e^{H_2} \le z(n) \le e^{H_1} = H_1', \\ H_4' &= e^{H_4} \le p(n) \le e^{H_3} = H_3'. \end{aligned} \tag{29}$$

Theorem 2 *Assume that*

(i) *there exist positive constant c and positive constants d_j $(j = 1, 2)$ such that*

$$\frac{d_1}{H_1'} - [d_1 - d_1 r(n)] - \frac{d_2 k(n)b(n)[C(n) + B(n)H_3'] + [d_2\theta(n) + d_1 b(n)]A(n)H_3'}{[C(n) + A(n)H_2' + B(n)H_4']^2} > c,$$

$$\frac{d_2}{H_3'} - \frac{d_2 k(n)b(n)B(n)H_1' - [d_1 b(n) + d_2\theta(n)][C(n) + A(n)H_1']}{[C(n) + A(n)H_2' + B(n)H_4']^2} > c;$$

(ii) *$H_1 > 0$, $H_2 > 0$, $H_3 > 0$, $H_4 > 0$ for all $n \in N$, which guarantee the system (7) is permanent*;

(iii) *$k(n)b(n)B(n) - \theta(n)A(n) > 0$, $[k(n)b(n)B(n) - \theta(n)A(n)]H_2' - \theta(n) > 0$ for $n \in N$.*

Then the positive periodic solutions of (7) *are globally stable.*

Proof Let $z^*(n)$, $p^*(n)$ be a positive periodic solution of system (7). In the following, we will prove that it is uniformly asymptotically stable. We introduce the change of variables

$$u_1(n) = z(n) - z^*(n), \qquad u_2(n) = p(n) - p^*(n).$$

System (7) can be transformed into

$$\begin{cases} u_1(n+1) = z(n)\exp\Big\{r(n)\big[1-z(n)\big] - \dfrac{b(n)p(n)}{A(n)z(n)+B(n)p(n)+C(n)}\Big\} \\ \qquad - z^*(n)\exp\Big\{r(n)\big(1-z^*(n)\big) - \dfrac{b(n)p^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)}\Big\}, \\ u_2(n+1) = p(n)\exp\Big\{k(n)\dfrac{b(n)p(n)}{A(n)z(n)+B(n)p(n)+C(n)} - D(n) - g_1\big(z(n), p(n)\big)\Big\} \\ \qquad - p^*(n)\exp\Big\{k(n)\dfrac{b(n)p^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)} - D(n) - g_1\big(z^*(n), p^*(n)\big)\Big\}, \end{cases} \tag{30}$$

which can be rewritten as

$$\begin{cases} u_1(n+1) = \exp\Big\{r(n)\big(1-z^*(n)\big) - \dfrac{b(n)p^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)}\Big\} \\ \qquad \times\Big\{\Big[1 - r(n)z^*(n) + \dfrac{b(n)A(n)p^*(n)z^*(n)}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2}\Big]u_1(n) \\ \qquad + \Big[\dfrac{-b(n)z^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)} + \dfrac{b(n)B(n)z^*(n)p^*(n)}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2}\Big]u_2(n) \\ \qquad + F_1\big(n, u(n)\big)\Big\} \\ u_2(n+1) = \exp\Big\{k(n)\dfrac{b(n)z^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)} - D(n) - g_1\big(z^*(n), p^*(n)\big)\Big\} \\ \qquad \times\Big\{\Big[\dfrac{k(n)b(n)p^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)} + \dfrac{-k(n)b(n)A(n)z^*(n)p^*(n) + \theta(n)A(n)[p^*(n)]^2}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2}\Big]u_1(n) \\ \qquad + \Big[C(n) + \dfrac{\theta(n)B(n)[p^*(n)]^2 - k(n)b(n)B(n)z^*(n)p^*(n)}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2} \\ \qquad - \dfrac{\theta(n)p^*(n)}{A(n)z^*(n)+B(n)p^*(n)+C(n)}\Big]u_2(n) + F_2\big(n, u(n)\big)\Big\}, \end{cases} \tag{31}$$

where $\|F_i(n,u)\|/\|u\|$ $(i = 1, 2)$ converges to zero as $\|u\| \to 0$.

In view of system (7), it follows from (31) that

$$\begin{cases} u_1(n+1) = z^*(n+1) \times \left\{\left[1 - r(n)z^*(n) + \dfrac{b(n)A(n)p^*(n)z^*(n)}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2}\right]\dfrac{u_1(n)}{z^*(n)} \right. \\ \qquad \left. + \left[\dfrac{-b(n)[C(n)+A(n)z^*(n)]}{[A(n)z^*(n)+B(n)p^*(n)+C(n)]^2}\right]u_2(n) + \dfrac{F_1(n,u(n))}{z^*(n)}\right\}, \\ u_2(n+1) = p^*(n+1) \times \left\{\left[\dfrac{k(n)b(n)[C(n)B(n)p^*(n)] + \theta(n)A(n)p^*(n)}{(C(n)+A(n)z^*(n)+B(n)p^*(n))^2}\right]u_1(n) \right. \\ \qquad + \left[C(n) + \dfrac{-k(n)b(n)B(n)z^*(n)p^*(n) - \theta(n)p^*(n)[C(n)+A(n)z^*(n)]}{(C(n)+A(n)z^*(n)+B(n)p^*(n))^2}\right]\dfrac{u_2(n)}{p^*(n)} \\ \qquad \left. + \dfrac{F_2(n,u(n))}{p^*(n)}\right\}. \end{cases} \tag{32}$$

Now, we can define the Lyapunov function V by

$$V\big(u(K)\big) = d_1\left|\frac{u_1(n)}{z^*(n)}\right| + d_2\left|\frac{u_2(n)}{p^*(n)}\right|,$$

where d_j $(j=1,2)$ are positive constants given in (i).

By calculating the difference of V along the solution of system (32) and using (iii), we can obtain

$$\begin{aligned} \Delta V \le & -\left\{\frac{d_1}{z^*(n)} - \big[d_1 - d_1 r(n)\big]\right. \\ & \left. - \frac{d_2 k(n)b(n)[C(n)+B(n)p^*(n)] + [d_2\theta(n) + d_1\alpha(n)]A(n)p^*(n)}{[C(n)+A(n)z^*(n)+B(n)p^*(n)]^2}\right\} z^*(n)\left|\frac{u_1(n)}{z^*(n)}\right| \\ & - \left\{\frac{d_2}{p^*(n)} - \frac{d_2 k(n)b(n)B(n)z^*(n) - [d_1 b(n) + d_2\theta(n)][C(n)+A(n)z^*(n)]}{(C(n)+A(n)z^*(n)+B(n)p^*(n))^2}\right\} p^*(n)\left|\frac{u_2(n)}{p^*(n)}\right| \\ & + d_1\left|\frac{F_1(n,u(n))}{z^*(n)}\right| + d_2\left|\frac{F_2(n,u(n))}{p^*(n)}\right|. \end{aligned} \tag{33}$$

In view of (29), we can obtain that

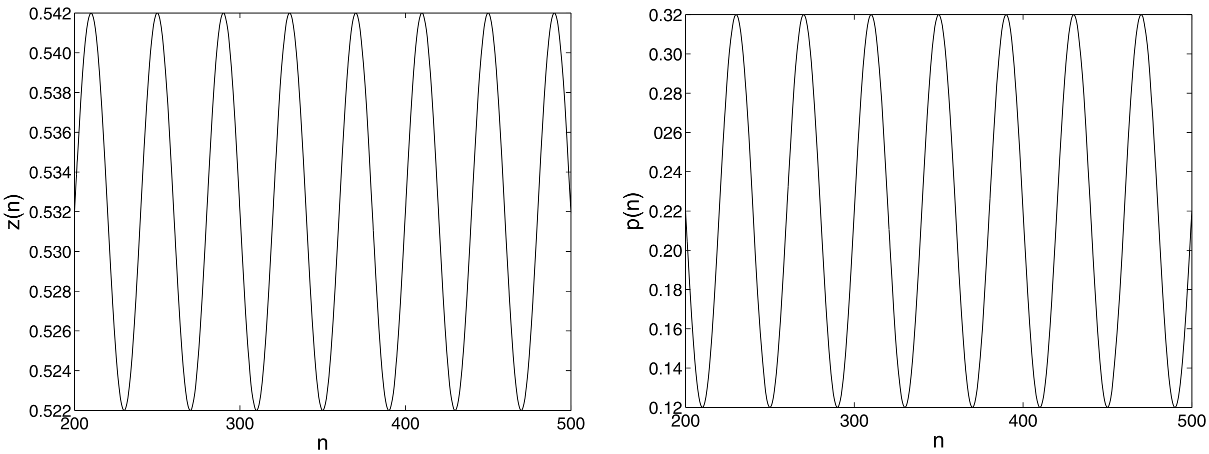
$$\begin{aligned} \Delta V \le & -\left\{\frac{d_1}{H_1'} - \big[d_1 - d_1 r(n)\big] - \frac{d_2 k(n)b(n)[C(n)+B(n)H_3'] + [d_2\theta(n)+d_1 b(n)]A(n)H_3'}{[C(n)+A(n)H_2'+B(n)H_4']^2}\right\} z^*(n)\left|\frac{u_1(n)}{z^*(n)}\right| \\ & - \left\{\frac{d_2}{H_3'} - \frac{d_2 k(n)b(n)B(n)H_1' - [d_1 b(n)+d_2\theta(n)][C(n)+A(n)H_1']}{[C(n)+A(n)H_2'+B(n)H_4']^2}\right\} p^*(n)\left|\frac{u_2(n)}{p^*(n)}\right| \\ & + d_1\left|\frac{F_1(n,u(n))}{z^*(n)}\right| + d_2\left|\frac{F_2(n,u(n))}{p^*(n)}\right|. \end{aligned} \tag{34}$$

Since $\|F_i(n,u)\|/\|u\|$ $(i=1,2)$ converges to zero as $\|u\| \to 0$, and in view of the assumptions (i) and (ii), there exists a positive constant δ such that if n is sufficiently large and $\|u(n)\| < \delta$,

$$\Delta V \le -\frac{c\|u(n)\|}{2} < -\frac{c\delta}{2}.$$

By [20], we see that the trivial solution of (30) is uniformly asymptotically stable, and so is the solution $(z^*(n), p^*(n))$ of (7).

We note that the positive solution $(z(n), p(n))$ is chosen in arbitrary way. As the proceeding exactly in [21], it can be concluded that the positive periodic



(a) (b)

Fig. 1 Positive periodic solution of system (7). Parameters values can be found in the main text

solutions $(z^*(n), p^*(n))$ of system (7) is globally stable.

The proof is completed. □

Example We choose $r(n) = 0.05[1 + \sin(\pi n/20)]$, $b(n) = 0.02\sin(\pi n/20)$, $A(n) = 0.1\sin(\pi n/20)$, $B(n) = 0.01\sin(\pi n/20)$, $C(n) = 0.3[2 + 0.1\sin(\pi n/20)]$, $k(n) = 0.002\sin(\pi n/20)$, $D(n) = 0.006\sin(\pi n/20)$, $\theta(n) = 0.01\sin(\pi n/20)$ for simulations. In Fig. 1, we found that system (7) has a positive periodic solution with $T = 40$.

5 Conclusions and discussions

In this paper, we present a discrete predator–prey system with the Beddington–DeAngelis functional response and predator cannibalism. We obtained the existence conditions of at least one periodic solution by using the continuation theorem. Moreover, by constructing a Lyapunov function, the sufficient conditions, which ensure the global stability of the positive periodic solutions, are gained. Biological speaking, if the populations exhibit periodic oscillation, they are prone to persist for a long time, which are beneficial for protecting the diversity of species.

All things live in a spatial world where it is a natural phenomenon that a substance goes from high density regions to low density regions [22–24]. Moreover, spatial models can be used to estimate the formation of spatial patterns on the large scale and guide policy decisions in the aspect of population conservation [25, 26]. As a result, dynamics of discrete predator–prey systems with space need further investigation and discussion.

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