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Fuzzy rough approximations for set-valued data

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ABSTRACT

Rough set theory is one of important tools of soft computing, and rough approximations are the essential elements in rough set models. However, the existing fuzzy rough set model for set-valued data, which is directly constructed based on a kind of similarity relation, fail to explicitly define fuzzy rough approximations. To solve this issue, in this paper, we propose two types of fuzzy rough approximations, and define two corresponding relative positive region reducts. Furthermore, two discernibility matrices and two discernibility functions are introduced to acquire these new proposed reducts, and the relationships among the new reducts and the existing reducts are also be provided. Theoretical analyses demonstrate that the new types of reducts have less redundancy and are more diverse (no lower number of reducts) than those obtained by means of the existing matrices, and experimental results illustrate the new reducts found by our methods outperform those obtained by existing method.

1. Introduction

Rough set was initially proposed by Pawlak [22,23,25], which is an important tool of soft computing research area. So far, it has become a popular mathematical framework for data mining and knowledge discovery [1,21,37]. Pawlak's rough set is modeled using an equivalence relation, and is only suitable to process categorical data. However, numerical attributes, or hybrid attributes (both categorical and numerical attributes) arise commonly in practice [10]. Thus, to solve this problem, researchers found two feasible ways. One is to transform numerical and hybrid attributes into categorical attributes by discretization [17,18]. But, this method could result in the information loss of original data. Therefore, the other way is to process numerical or hybrid data by the fuzzy rough set [5,15,16,28,35,40].

Fuzzy rough sets encapsulate the related but distinct concepts of vagueness and indiscernibility [2,30]. It was first proposed by Dubois and Prade [5,6]. Whereafter, researchers generalized fuzzy rough set models by the constructive and axiomatic approaches. Radzikowska and Kerre [27] presented a more general fuzzy rough set model through employing t-norm and t-conorm in lower and upper approximations. Mi and Zhang [14] introduced a new fuzzy rough set definition based on a residual implication θ and its dual σ. Yeung et al. [39] proposed some fuzzy rough set models by means of arbitrary fuzzy relations and investigated the connections between the existing fuzzy rough sets. Hu et al. [9] proposed a novel fuzzy rough set model, based on which a simple and efficient hybrid attribute reduction algorithm was designed. Wang et al. [32] defined new lower and upper approximations based on the similarity between two objects and extended some underlying concepts to the fuzzy environment. Tsang et al. [30] introduced formal concepts of attributes reduction by means of fuzzy rough sets

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and give the structure of attribute reduction. Wei et al. [33] investigated the relationships among rough approximations of fuzzy rough set models. Chen et al. [3] gave the interpretation of several types of membership functions geometrically by using the lower approximations in fuzzy rough sets, in terms of square distances in Krein spaces. Yao et al. [38] presented a variable precision (θ, σ)-fuzzy rough set model based on fuzzy granules, which is more robust than the existing models. These fuzzy rough set models mentioned above are the generalization of classical rough set model, and they are employed to effectively process numerical data and hybrid data [33].

In many practical applications, a set may appear in some of the attribute values for an object. An information system with set-valued attributes, which is commonly called set-valued information system, just represents the set-valued data very well. Orlowska and Pawlak [19,20] investigated set-valued information systems considering non-deterministic information and introduced the concept of a non-deterministic information system. Yao and Noroozi [36] proposed a number of set-based computation methods, set-valued information systems as one of these models was explicitly introduced. Guan and Wang [7] proposed a rough set model based on a tolerance relation derived from set-valued information systems. Qian et al. [26] first introduced a set-valued ordered information system, and to process the information system a rough set model was proposed. Zhang et al. [41] introduced several approaches for updating the lower and upper approximations in a set-valued information system. Moreover, set-valued information systems can be employed to handle incomplete data through the set of all possible values for each attribute replacing missing values in an incomplete information system [8]. In all, some significant research work about set-valued data has been done. However, so far, only a few studies have focused on fuzzy rough set model for set-valued data.

To process set-valued data by fuzzy rough set model, Dai and Tian [4] defined a fuzzy similar relation which retains more information than the existing crisp similarity relation does. However, fuzzy rough approximations for set-valued data was not explicitly defined. Though the fuzzy similarity relation is important to construct rough approximations, it does not replace fuzzy rough approximations that are the key components for a rough set model. To solve this problem, in this paper, we will propose two new types of fuzzy lower and upper approximations, and further define two discernibility matrices and two discernibility functions will be introduced which is used to obtain the two new types of reducts, i.e. Type-1 and Type-2 reducts. Sequently, we will present the relationships among the two types of reducts and the existing reduct in [4]. Theoretical analyses and experimental results demonstrate that the new proposed reducts have less redundancy and are more diverse (no lower number of reducts) than those obtained by the existing matrices, and the newfound reducts outperform those by existing method from the perspective of coverage degree of rules and percentage of correct classification.

The rest of this paper is organized as follows. Some preliminary concepts are briefly reviewed in Section 2. In Section 3, two new types of lower and upper approximations and the corresponding positive regions are proposed. In Section 4, we define two new types of reducts in the sense of positive region, construct the two discernibility matrices to obtain these reducts, and investigate the relationships between the proposed reducts with Dai's reducts. In Section 5, we carry out several numerical experiments to verify the theoretical results. Section 6 concludes this paper.

2. Preliminaries

2.1. Rough set model for Set-valued information systems

A set-valued information system provides a convenient framework for the representation of the objects described with set-valued attributes. Let $SIS = (U, A, V, f)$ be a set-valued information system, where U is a non-empty and finite set of objects, called a universe, and A is a non-empty and finite set of attributes with set values. For each $a \in A$, a mapping f: $U \to V_a$ is determined by a set-valued information system, where V_a is the domain of a. Additionally, $f(a, x)$ indicates the value of x on attribute a, and it will be represented by $a(x)$ for brevity hereafter.

Base on a set-valued information system $SIS = (U, A, V, f)$, a similarity relation derived from $a \in A$ can be constructed as follow:

$$R_a = \{(x_i, x_j) | a(x_i) \cap a(x_j) \neq \emptyset\}.$$

And, we can define the corresponding similarity relation derived from $B \subseteq A$ as

$$R_B = \{(x_i, x_j) | a(x_i) \cap a(x_j) \neq \emptyset, \forall a \in B\}.$$

The similar relation can also be represented by the following matrix

$$R_B = \begin{pmatrix} r_{11}^B & r_{12}^B & \cdots & r_{1j}^B & \cdots & r_{1n}^B \\ r_{21}^B & r_{22}^B & \cdots & r_{2j}^B & \cdots & r_{2n}^B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{i1}^B & r_{i2}^B & \cdots & r_{ij}^B & \cdots & r_{in}^B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r_{n1}^B & r_{n2}^B & \cdots & r_{nj}^B & \cdots & r_{nn}^B \end{pmatrix},$$

where $r_{ij}^B = 1$ if $a(x_i) \cap a(x_j) \neq \emptyset$ *for* $\forall a \in B$, otherwise $r_{ij}^B = 0$, and $n = |U|$.

Based on the fuzzy similarity relation mentioned above, the similarity class of x with respect to $B(B \subseteq C)$ is defined as

$$S_B(x_i) = \{x_j | (x_i, x_j) \in R_B, \forall x_j \in U\},$$

when x_i and x_j are indiscernible with respect to B, or x_i is similar to x_j with respect to B.

For any $Y \subseteq U$, one defines that $(\overline{B}(Y), \underline{B}(Y))$ is the rough set of Y with respect to B, where the lower approximation $\underline{B}(Y)$ and the upper approximation $\overline{B}(Y)$ of Y [7] are described by

$$\underline{B}(Y) = \{x | S_B(x) \subseteq Y\}, \text{ and}$$
$$\overline{B}(Y) = \{x | S_B(x) \cap Y \neq \emptyset\}.$$

The objects in $\underline{B}(Y)$ can be certainly classified as members of Y by the knowledge in B, while the objects in $\overline{B}(Y)$ can be only classified as possible members of Y by the knowledge in B. The set $BN_B(Y) = \overline{B}(Y) - \underline{B}(Y)$ is called the $B-$ boundary region of Y, and thus consists of those objects that we cannot decisively classify into Y on the basis of the knowledge in B [11].

To represent a classification problem with set-valued condition attributes, we introduce a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. In a set-valued decision table, C is called a condition attribute set, $\{d\}$ is called a decision attribute, and $C \cap \{d\} = \emptyset$. In this paper, we focus on the set-valued decision table in which the values of condition attributes are set-valued and the values of decision attributes are categorical.

In a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, the lower and upper approximations of $\{d\}$ with respect to B are defined as [4]

$$\underline{B}(\{d\}) = \cup\{x | S_B(x) \subseteq Y_i, Y_i \in U/\{d\}\}, \text{ and}$$
$$\overline{B}(\{d\}) = \cup\{x | S_B(x) \cap Y_i \neq \emptyset, Y_i \in U/\{d\}\}.$$

The relative positive region of $\{d\}$ with respect to B is defined as $POS_B(\{d\}) = \cup_{Y_i \in U/D} \underline{B}(Y_i)$, where $Y_i \in U/\{d\}$ and $U/\{d\}$ represents the set of equivalence classes of partitioning U by $\{d\}$.

2.2. Similarity relation derived from set-valued information systems

For set-valued data, a similarity relation, which is derived from the similar degree between two objects, plays a critical role of constructing fuzzy rough set models. In [4], a similar degree in set-valued data is defined as follows.

Definition 2.1 [4]. Given the set-valued information system $SIS = (U, A, V, f)$, $\forall a \in A$, a similar degree $\widetilde{r}_{ij}^a$ between x_i and x_j is defined as

$$\widetilde{r}_{ij}^a = \frac{|a(x_i) \cap a(x_j)|}{|a(x_i) \cup a(x_j)|},$$

and for a set of attribute $B \subseteq A$, a fuzzy relation R_B is defined as $\widetilde{r}_{ij}^B = \min_{a \in B}\{\widetilde{r}_{ij}^a\}$.

Based on the similar degree, a fuzzy similarity relation is defined as

$$\widetilde{R}_B = \begin{pmatrix} \widetilde{r}_{11}^B & \widetilde{r}_{12}^B & \cdots & \widetilde{r}_{1j}^B & \cdots & \widetilde{r}_{1n}^B \\ \widetilde{r}_{21}^B & \widetilde{r}_{22}^B & \cdots & \widetilde{r}_{2j}^B & \cdots & \widetilde{r}_{2n}^B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \widetilde{r}_{i1}^B & \widetilde{r}_{i2}^B & \cdots & \widetilde{r}_{ij}^B & \cdots & \widetilde{r}_{in}^B \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \widetilde{r}_{n1}^B & \widetilde{r}_{n2}^B & \cdots & \widetilde{r}_{nj}^B & \cdots & \widetilde{r}_{nn}^B \end{pmatrix},$$

where $n = |U|$.

2.3. Fuzzy rough set models

Dubois and Prade first introduced the fuzzy rough set [5], hereafter called Dubois' fuzzy rough set for simplicity. According to their definition, a universe of objects $U = \{x_1, x_2, \ldots, x_n\}$ is described by a fuzzy binary relation $\widetilde{R}$, and the membership of object x_i in a fuzzy rough set $(\underline{\widetilde{R}}(X), \overline{\widetilde{R}}(X))$ is described as

$$\mu_{\underline{\widetilde{R}}(X)}(x_i) = \inf_{x_j \in U} \max\{1 - \widetilde{R}(x_i, x_j), \mu_X(x_j)\} \text{ and}$$

$$\mu_{\overline{\widetilde{R}}(X)}(x_i) = \sup_{x_j \in U} \min\{\widetilde{R}(x_i, x_j), \mu_X(x_j)\},$$

where $X \in \mathscr{F}(U)$. $\mathscr{F}(U)$ is the class of all fuzzy sets in U.

Radzikowska and Kerre presented a more generalized fuzzy rough set model [27]. They introduced a broad family of fuzzy rough sets, each called a $(\mathscr{I}, \mathscr{T})$-fuzzy rough set, determined by an implicator $\mathscr{I}$ and triangular norm $\mathscr{T}$. The corresponding fuzzy approximation space and fuzzy rough approximations are defined below.

Let $S = (U, \widetilde{R})$ be a fuzzy approximation space, U be the discoursed universe and $\widetilde{R}$ be a fuzzy similarity relation on U. Let $\mathscr{I}$ and $\mathscr{T}$ be a border implicator and a t-norm, respectively. The $(\mathscr{I}, \mathscr{T})$-fuzzy rough approximation in S is a mapping $Apr_S^{\mathscr{I},\mathscr{T}} : \mathscr{F}(U) \to \mathscr{F}(U) \times \mathscr{F}(U)$, which is defined as
for every $X \in \mathscr{F}(U)$

$$Apr_S^{\mathscr{I},\mathscr{T}} = ((\widetilde{R} \downarrow X)_{\mathscr{I}}(x_i), (\widetilde{R} \uparrow X)^{\mathscr{T}}(x_i)),$$

and for every $x_i \in U$

$$(\widetilde{R} \downarrow X)_{\mathscr{I}}(x_i) = \inf_{x_j \in U} \mathscr{I}(\widetilde{r}_{ij}, \mu_X(x_i)),$$

$$(\widetilde{R} \uparrow X)^{\mathscr{T}}(x_i) = \sup_{x_j \in U} \mathscr{T}(\widetilde{r}_{ij}, \mu_X(x_i)),$$

where $\mathscr{F}(U)$ is the class of all fuzzy sets of U.

For explicitly indicating the above fuzzy rough set model, we briefly review the implicator and t-norm as follows.

A triangular norm, or t-norm, is an increasing, associative and commutative mapping $\mathscr{T} : [0,1]^2 \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], T(x,1) = x)$. The standard min operator $\mathscr{T}_M(x,y) = \min\{x,y\}$, the algebraic product $\mathscr{T}_P(x,y) = x * y$, and the bold intersection (also called the Łukasiewicz t-norm) $\mathscr{T}_L(x,y) = \max\{0, x+y-1\}$ are the most popular continuous t-norms.

A triangular conorm, or $t-$ conorm, is an increasing, associative and commutative mapping $\mathscr{S} : [0,1]^2 \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], \mathscr{S}(x,0) = x)$. The standard max operator $\mathscr{S}_M(x,y) = \max\{x,y\}$ (the smallest t-conorm), the probabilistic sum $\mathscr{S}_P(x,y) = x + y - x * y$, and the bounded sum $\mathscr{S}_L(x;y) = \min\{1, x+y\}$ are the most popolar continuous conorms.

A negator $\mathscr{N}$ is a decreasing $[0,1] - [0,1]$ mapping satisfying $\mathscr{N}(0) = 1$ and $\mathscr{N}(1) = 0$. The negator $\mathscr{N}_s = 1 - x$ is usually referred to as the standard negator. A negator $\mathscr{N}$ is involutive if $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in [0, 1]$, and it is weakly involutive if $\mathscr{N}(\mathscr{N}(x)) \geq x$ for all $x \in [0, 1]$.

Let $\mathscr{T}, \mathscr{S}$ and $\mathscr{N}$ be a t-norm, t-conorm and negator, respectively. An implicator $\mathscr{I}$ is called an S-implicator based on $\mathscr{S}$ and $\mathscr{N}$ if $\mathscr{I}(x,y) = \mathscr{T}(\mathscr{N}(x), y)$ for all $x, y \in [0, 1]$. The Łukasiewicz implicator $\mathscr{I}_L(x,y) = min\{1, 1-x+y\}$, based on $\mathscr{S}_L$ and $\mathscr{N}_s$, the Kleene-Dienes implicator $\mathscr{I}_{KD}(x,y) = \max\{1-x, y\}$, based on $\mathscr{S}_M$ and $\mathscr{N}_s$, the Kleene-Dienes-Łukasiewicz implicator $\mathscr{I}_\star(x,y) = 1 - x + x * y$, based on $\mathscr{S}_P$ and $\mathscr{N}_s$ are the most popular S-implicators.

When X is a crisp subset of U, both Dubois' fuzzy rough approximations and Radzikowska's fuzzy rough approximations are simplified as

$$\mu_{\underline{C}(X)}(x_i) = \min_{x_j \notin X}\{1 - \widetilde{r}_{ij}\}, x_i \in X, \text{and}$$

$$\mu_{\overline{C}(X)}(x_i) = \max_{x_j \in X}\{\widetilde{r}_{ij}\}.$$

In [9], Hu et al. presented another fuzzy rough approximations based on the fuzzy information granules, which are defined as follows:

$$\underline{HC}(X) = \{x_i | [x_i]_C \subseteq X,\ x_i \in U\}, \text{ and}$$
$$\overline{HC}(X) = \{x_i | [x_i]_C \cap X \neq \emptyset,\ x_i \in U\},$$

where $[x_i]_C$ is a fuzzy information granule of x with respect to attribute set C.

For a given decision table $S = (U, C \cup \{d\}, V, f)$, $U/\{d\} = \{Y_1, Y_2, \ldots, Y_N\}$ is a partition of discoursed universe U. The lower and upper approximations with respect to D are

$$\underline{HC}(\{d\}) = \cup_{i=1}^{N} \underline{HC}(Y_i), \text{and}$$
$$\overline{HC}(\{d\}) = \cup_{i=1}^{N} \overline{HC}(Y_i).$$

3. Fuzzy rough approximations for Set-valued data

Lower and upper approximations are crucial for a rough set model. In this section, we will define two fuzzy rough approximations for set-valued data based on the similar degree defined in Section 2.2. The first type of fuzzy rough lower and upper approximations are constructed by using a fuzzy similarity class of each object. The second type of fuzzy rough lower and upper approximations are constructed by using a similar degree between two objects.

To define the first type of fuzzy rough approximations, we define the similarity class in a similar way in Ref. [9].

Definition 3.1. Given the set-valued information system $SIS = (U, A, V, f)$, a fuzzy similarity class of x_i with respect to the fuzzy similar relation $\widetilde{R}_B$ is defined as

$$\widetilde{S}_B(x_i) = \sum_{j=1}^{n} \frac{\widetilde{r}_{ij}^B}{x_i},$$

where $\widetilde{r}_{ij}^B$ is a similar degree between two objects in U.

Furthermore, the relationship between fuzzy similarity class and crisp similarity class will be described by the following theorem.

Theorem 3.1. *Given the set-valued information system* $SIS = (U, A, V, f)$, $S_B(x_i)$ *and* $\widetilde{S}_B(x_i)$ *are crisp and fuzzy similarity classes of* x_i*, respectively. Then the relationship between them is*

$$(\widetilde{S}_B(x_i))_0 = S_B(x_i),$$

where $(\widetilde{S}_B(x_i))_0$ *represents the 0-cut of* $\widetilde{S}_B(x_i)$.

Proof.

$$\begin{aligned}(\widetilde{S}_B(x_i))_{\alpha=0} &= \{x_j | \widetilde{r}_{ij}^B > 0, 1 \le j \le n\}\\ &= \{x_j | \widetilde{r}_{ij}^a > 0, \forall a \in B, 1 \le j \le n\}\\ &= \{x_j | a(x_i) \cap a(x_j) \ne \emptyset, \forall a \in B, 1 \le j \le n\}\\ &= S_B(x_i).\end{aligned}$$

□

From Theorem 3.1, it is easy to know that a 0-*cut* of fuzzy similarity class is identical to the classical similarity class in [7].

Based on the fuzzy information granule mentioned above, we introduce the first type of fuzzy rough lower and upper approximations in the following definition.

Definition 3.2. Given a set-valued information system $SIS = (U, A, V, f)$, $B \subseteq A$ and $X \subseteq U$. Then Type-1 lower and upper approximations of X with respect to B are defined as

$$\underline{app}_B^1(X) = \{x_i | \widetilde{S}_B(x_i) \subseteq X, X \subseteq U\}, \text{ and}$$
$$\overline{app}_B^1(X) = \{x_i | \widetilde{S}_B(x_i) \cap X \ne \emptyset, X \subseteq U\}.$$

If the fuzzy similarity class degrades to the crisp similarity class in a set-valued information system, Type-1 fuzzy rough lower and upper approximations also degrade to the crisp ones, which is indicated in the following theorem.

Theorem 3.2. *Given a set-valued information system* $SIS = (U, A, V, f)$. *If* $\widetilde{S}_B(x_i)$ *is a crisp set, then*

$$\underline{app}_B^1(X) = \underline{B}(Y), \text{ and}$$
$$\overline{app}_B^1(X) = \overline{B}(Y).$$

The theorem is easy to be proved by the conclusion of Theorem 3.1. Therefore, we omit the proof.

Furthermore, for a set-valued decision table, we also define the corresponding Type-1 fuzzy rough lower and upper approximations.

Definition 3.3. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, Type-1 fuzzy rough lower upper approximations of $\{d\}$ with respect to B are defined as

$$\underline{app}_B^1(\{d\}) = \{x_i | \widetilde{S}_B(x_i) \subseteq Y, Y \in U/\{d\}\}, \text{ and}$$
$$\overline{app}_B^1(\{d\}) = \{x_i | \widetilde{S}_B(x_i) \cap Y \ne \emptyset, Y \in U/\{d\}\}.$$

Furthermore, the positive region of $\{d\}$ with respect to B ($B \subseteq C$) in the set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$ is defined as

$$POS_B^1(\{d\}) = \underline{app}_B^1(\{d\}).$$

We regard the set of the objects that belong to the positive region as the consistent part of a set-valued decision table, and the set of the other objects as the inconsistent part of the set-valued decision table.

Sequently, another type of lower and upper approximations are defined in the following definition.

Definition 3.4. Given a set-valued information system $SIS = (U, A, V, f)$, $B \subseteq A$ and $X \subseteq U$, then the membership function of X_i with respect to Type-2 fuzzy rough lower and upper approximations are defined as

$$\mu_{\underline{\widetilde{app}}_B^2(X)}(x_i) = \min_{x_j \notin X}\{1 - \widetilde{r}_{ij}^B\} \text{ and}$$
$$\mu_{\overline{\widetilde{app}}_B^2(X)}(x_i) = \max_{x_j \in X}\{\widetilde{r}_{ij}^B\},$$

where $\widetilde{r}_{ij}^B$ is the similar degree between $x_i \in U$ and $x_j \in U$ on B.

Furthermore, for a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, the membership function of $x_i \in U$ with respect to Type-2 positive region is defined as

$$\mu_{\widetilde{POS}_B^2(\{d\})}(x_i) = \max_{Y \in U/\{d\}}\{\mu_{\underline{\widetilde{app}}_B^2(Y)}(x_i)\}.$$

The Type-1 fuzzy rough lower and upper approximations are constructed based on a fuzzy similarity class, and the Type-2 fuzzy rough lower and upper approximations are constructed based on a fuzzy similar degree. From the definition of the fuzzy similarity class, it is easy to know that the fuzzy similarity class is the fuzzy set on which the membership function of each object is the similar degree between the objects. Therefore, we can infer that there are some relationships between them, and we will investigate these issues in the following theorems.

Theorem 3.3. *Given a set-valued information system $SIS=(U,A,V,f)$, $B\subseteq A$ and $X\subseteq U$, then the relationship between Type-1 fuzzy rough lower approximation and Type-2 rough lower approximation is*

$$\underline{app^1}_B(X)=(\underline{\widetilde{app}^2}_B(X))_1,$$

where $(\underline{\widetilde{app}^2}_B(X))_1$ is the 1-cut set of the fuzzy set $\underline{\widetilde{app}^2}_B(X)$.

Proof. By the existing conditions, we have that

$$\begin{aligned}(\underline{\widetilde{app}^2}_B(X))_1 &= \{x_i|\underline{\widetilde{app}^2_B}(X)(x_i)\ge 1\}\\ &= \{x_i|\min_{x_j\notin X}\{1-\tilde{r}^B_{ij}\}\ge 1\}\\ &= \{x_i|(1-\tilde{r}^B_{ij})\ge 1,\forall x_j\notin X\}\\ &= \{x_i|\tilde{r}^B_{ij}=0,\forall x_j\notin X\}\\ &= \{x_i|\tilde{S}_B(x_i)\subseteq X, X\subseteq U\}\\ &= \underline{app^1}_B(X).\end{aligned}$$

□

From Theorem 3.3, we can see that the Type-1 fuzzy rough low approximation is a special case of the Type-2 fuzzy rough low approximation. And, by the results of Theorem 3.3, we can infer a relationship between the Type-1 relative positive region and the Type-2 relative positive region as follows.

Corollary 3.1. *Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, $B\subseteq A$, then*

$$POS^1_B(\{d\})=(\widetilde{POS}^2_B(\{d\}))_1,$$

where $(\widetilde{POS}^2_B(\{d\}))_1$ is the 1-cut set of $\widetilde{POS}^2_B(\{d\})$.

Proof. From the definition of Type-2 positive region, we have $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=\max_{Y\in U/\{d\}}\{\mu_{\underline{\tilde{R}}(Y)}(x_i)\}=\max_{Y_k\in U/\{d\}}\{\min_{x_j\notin Y_k}\{1-\tilde{r}^B_{ij}\}\}$. For Y_k ($Y_k\in U/\{d\}$) containing x_i, $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=\max_{Y_k\in U/\{d\}}\{\min_{x_j\notin Y_k}\{1-\tilde{r}^B_{ij}\}\}$, and for Y_k not containing x_i, $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=0$, we thus have that $\min_{x_j\notin Y_k}\{1-\tilde{r}^B_{ij}\}$ gets the maximum if $x_i\in Y_k$. Furthermore, $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=\min_{x_j\notin Y_k}\{1-\tilde{r}^B_{ij}\}$ if $x_i\in Y_k$, and $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=0$ if $x_i\notin Y_k$ (i.e. $\mu_{\widetilde{POS}^2_B(\{d\})}(x_i)=\mu_{\underline{\widetilde{app}^2}_B(Y_k)}(x_i)$ where $x_i\in Y_k$). Finally, by means of Theorem 3.3, it is easy to obtain $POS^1_B(\{d\})=(\widetilde{POS}^2_B(\{d\}))_1$. □

From Corollary 3.1, we can see that the Type-1 relative positive region is the special case of the Type-2 positive region.

4. Attribute reduction based on discernibility matrix

Discernibility matrix is one of important issues in rough set theory [29,34], by which all reducts of condition attribute set with respect to decision attribute can be obtained. In this section, we first propose two new types of discernibility matrices. By them, we can get attribute reducts of preserving Type-1 positive region and Type-2 positive region, respectively. And then, we investigate relationships among Type-1 positive region reducts, Type-2 positive region reducts and Dai's reducts. For the development of this section, the definition of attribute reduct in [4] is first reviewed.

Definition 4.1 [4]**.** Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, $B\subseteq C$. B is a Dai's reduct of C if and only if

(1) $\forall x_i$, $x_j\in U$, if $d(x_i)\neq d(x_j)$, then $\tilde{r}^B_{ij}=\tilde{r}^C_{ij}$;

(2) For any $B'\subset B$, $\exists x_i$, $x_j\in U$, $d(x)\neq d(y)$ and $\tilde{r}^B_{ij}\neq\tilde{r}^C_{ij}$.

To compute Dai's reducts in Definition 4.1, a discernibility matrix was defined in [4], which is showed in the following definition.

Definition 4.2 [4]**.** Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$. Then Dai's discernibility matrix is defined as $M^{Dai}_{n\times n}=\{m^{Dai}_{ij}\}$, where

$$m^{Dai}_{ij}=\begin{cases}\{a\in C:\tilde{r}^a_{ij}=\tilde{r}^C_{ij}\}, & d(x_i)\neq d(x_j)\\ \emptyset, & d(x_i)=d(x_j)\end{cases}.$$

Based on the discernibility matrix, a discernibility function can be defined as follow.

Definition 4.3 [4]. Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$. A discernibility function f_{Dai} for SDT is a Boolean function of m Boolean variables $c_1^*,c_2^*,\ldots,c_m^*$ corresponding to the attribute $c_1,c_2,\ldots,c_m$, respectively, and defined as

$$f_{Dai}(c_1^*,c_2^*,\ldots,c_m^*)=\wedge\{\vee m_{ij}^{Dai}\in M_{n*n}^{Dai},\ m_{ij}^{Dai}\neq\emptyset\},$$

where $\vee m_{ij}^{Dai}$ is the disjunction of all variables c^* such that $a\in m_{ij}^{Dai}$ and $\wedge$ denotes conjunction.

The discernibility function $f_{Dai}(c_1^*,c_2^*,\ldots,c_m^*)$ describes the constraints which must hold to preserve similar degree between two objects from SDT. For a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, the set of all prime implicants of $f_{Dai}(c_1^*,c_2^*,\ldots,c_m^*)$ determines the set of all reducts of SDT.

In addition, Ref. [4] gives the definition of core of condition attribute set with respect to decision attribute. We rewrite the definition in the following definition.

Definition 4.4. Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, then Dai's core of C with respect to $\{d\}$ is defined as

$$Core_{\{d\}}^{Dai}(C)=\cap_{B\in\mathbf{RED}_{\{d\}}^{Dai}(C)}B,$$

where $\mathbf{RED}_{\{d\}}^{Dai}(C)$ is the set of all reducts of preserving similar degree between two objects.

From the definition of Dai's reduct, we can see that the reduct preserves the similar degree between each pair-wise objects after reducing a set-valued decision table. However, the discernibility of preserving similar degree is so strict that it leads to many redundant attributes in reducts. To solve this problem, we will propose two types of new discernibility matrixes. In the following, we introduce the first type of reduct that preserve the Type-1 positive region.

Definition 4.5. Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, B ($B\subseteq C$) is a Type-1 positive region reduct of C with respect to $\{d\}$ if and only if

(1) $\forall x,y\in U, POS_C^1(\{d\})=POS_B^1(\{d\})$;

(2) For any $B'\subset B, POS_B^1(\{d\})\neq POS_{B'}^1(\{d\})$.

To obtain the defined a Type-1 positive region reduct, we give a new discernibility matrix in the following definition.

Definition 4.6. Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$. Then Type-1 discernibility matrix is defined as $M_{n\times n}^1=\{m_{ij}^1\}$, where

$$m_{ij}^1=\begin{cases}\{a\in C:\widetilde{r}_{ij}^a=\min_{b\in C}\{\widetilde{r}_{ij}^b\}, & d(x_i)\neq d(x_j),\ \text{and } x_i,x_j\in U_1\\ \{a\in C:\widetilde{r}_{ij}^a=\min_{b\in C}\{\widetilde{r}_{ij}^b\}, & d(x_i)\neq d(x_j),\ \text{and } x_i\in U_1,x_j\in U_2\\ \emptyset, & \text{otherwise}\end{cases},$$

U_1 is the consistent part of the decision table SDT, and U_2 is the inconsistent part of the decision table SDT.

In the following, a theorem is employed to illustrate the relationship between a Type-1 positive region reduct and the element in the Type-1 discernibility matrix derived from SDT, which is the theoretical fundament that assures all Type-1 positive region reducts can be obtained by a Type-1 discernibility matrix.

Theorem 4.1. *Given a set-valued decision table $SDT=(U,C\cup\{d\},V,f)$, then B ($B\subseteq C$) is a Type-1 positive region reduct of C if and only if B is the minimal set satisfying $B\cap m_{ij}^1\neq\emptyset$ for $\forall m_{ij}^1\neq\emptyset$.*

Proof. ($\Leftarrow$) If $d(x_i)\neq d(x_j)$ and x_i, $x_j\in U_1$, or $d(x_i)\neq d(x_j)$ and $x_i\in U_1$, $x_j\in U_2$, or $d(x_i)\neq d(x_j)$ and $x_i\in U_2$, $x_j\in U_1$, then $m_{ij}^1=\{a\in C:\widetilde{r}_{ij}^a=\min_{b\in C}\{\widetilde{r}_{ij}^b\}$.

By the existing condition, we have that $B\cap m_{ij}^1\neq\emptyset$ for $\forall m_{ij}^1\neq\emptyset$. Without any loss of generality, let $a\in B\cap m_{ij}^1$, then $a\in m_{ij}^1$ and $a\in B$. By Definition 4.5, we have that $\widetilde{r}_{ij}^a=\min_{b\in C}\{\widetilde{r}_{ij}^b\}$, if $d(x_i)\neq d(x_j)$ and x_i, $x_j\in U_1$, or and $x_i\in U_1$ and $x_j\in U_2$, or and $x_i\in U_2$ and $x_j\in U_1$. Because of $a\in B$, $\widetilde{r}_{ij}^B=\min_{b\in B}\{\widetilde{r}_{ij}^b\}$ and $B\subseteq C$, $\widetilde{r}_{ij}^B=\min_{b\in B}\{\widetilde{r}_{ij}^b\}=\widetilde{r}_{ij}^a=\min_{b\in C}\{\widetilde{r}_{ij}^b\}=\widetilde{r}_{ij}^C$, if $d(x_i)\neq d(x_j)$ and x_i, $x_j\in U_1$, or and $x_i\in U_1$ and $x_j\in U_2$. Moreover, by Definition 4.5, it is easy to obtain $m_{ij}^1=\emptyset$ when x_i, $x_j\in U_2$, thus $B\cap m_{ij}^1=\emptyset$. Then, $\widetilde{r}_{ij}^B\geq\min_{b\in C}\{\widetilde{r}_{ij}^b\}=\widetilde{r}_{ij}^C$ in this case.

Furthermore, without any loss of generality, we suppose $x_i\in Y_k$. It is easy to know that $0\leq\widetilde{r}_{ij}^C\leq r_{ij}^d=1$ for $\forall x_j\in Y_k$, and $\widetilde{r}_{ij}^C=r_{ij}^d=0$ for $\forall x_j\notin Y_k$. For the development of the proof, we divide two cases as follows.

(1) $x_i\in U_1$

In this case, it is evident that $x_i\in POS_C^1(\{d\})$. And, without any loss of generality, we suppose $x_i\in Y_k$. Then, it is easy to know $\widetilde{r}_{ij}^C=r_{ij}^d=0$ for $\forall x_j\notin Y_k$. Furthermore, by the conclusion $\widetilde{r}_{ij}^B=\widetilde{r}_{ij}^C$ if x_i, $x_j\in U_1$, or $x_i\in U_1$ and $x_j\in U_2$, we have that $x_i\in POS_B^1(\{d\})$.

(2) $x_i\in U_2$

In this case, it is evident that x_i does not belong $POS_C^1(\{d\})$. Without any loss of generality, we suppose $x_i \in Y_k$, then it is easy to know $\widetilde{r}_{ij}^C = r_{ij}^d = 0$ for $\forall x_j \notin Y_k$ and $x_j \in U_1$, and $\exists x_j \notin Y_k$ and $x_j \in U_2$ such that $\widetilde{r}_{ij}^C > 0$. Furthermore, by the conclusion $\widetilde{r}_{ij}^B = \widetilde{r}_{ij}^C$ if $x_i \in U_2$ and $x_j \in U_1$ and the conclusion $\widetilde{r}_{ij}^B \geq \widetilde{r}_{ij}^C$, we have that $\widetilde{S}_B(x_i) \supseteq \widetilde{S}_C(x_i)$ and $x_i \notin POS_B^1(\{d\})$.

From the analysis of the two cases mentioned above, we have that for $\forall x_i \in U$, $x_i \in POS_B^1(\{d\})$ if $x_i \in POS_C^1(\{d\})$ and $x_i \notin POS_B^1(\{d\})$ if $x_i \notin POS_C^1(\{d\})$. Thus, $POS_B^1(\{d\}) = POS_C^1(\{d\})$.

Since B is the minimal set satisfying $B \cap m_{ij}^1 \neq \emptyset$ for every $m_{ij}^1 \neq \emptyset$, we know that for $\forall B' \subset B, \exists m_{pq}^1 \neq \emptyset$ such that $B' \cap m_{pq}^1 = \emptyset$, where $d(x_p) \neq d(x_q)$, and $x_p, x_q \in U_1$, or and $x_p \in U_1$ and $x_q \in U_2$, or and $x_p \in U_2$ and $x_q \in U_1$. Furthermore, combination with Definition 4.5, we have that for $\forall a \in B'$, $\widetilde{r}_{pq}^a > \min_{b \in C}\{\widetilde{r}_{pq}^b\}$. For the development of the proof, we divide three cases as follows.

(1) $x_p \in U_1$, $x_q \in U_1$

In this case, it is evident that x_p belongs to $POS_C^1(\{d\})$ and x_q belongs to $POS_C^1(\{d\})$. Without any loss of generality, we suppose $x_p \in Y_k$ and $x_q \notin Y_k$, then $\widetilde{r}_{pq}^C = r_{pq}^d = 0$. By the conclusion $\widetilde{r}_{pq}^{B'} > \widetilde{r}_{pq}^C$ if $x_p, x_q \in U_1$, we have that $\widetilde{S}_{B'}(x_p) \supseteq \widetilde{S}_C(x_p)$, $\widetilde{S}_{B'}(x_q) \supseteq \widetilde{S}_C(x_q)$. Therefore, $x_p \notin POS_{B'}^1(\{d\})$ and $x_q \notin POS_{B'}^1(\{d\})$.

(2) $x_p \in U_1$, $x_q \in U_2$

In this case, it is evident that x_p belongs to $POS_C^1(\{d\})$ and x_q does not $POS_C^1(\{d\})$. Without any loss of generality, we suppose $x_p \in Y_k$ and $x_q \notin Y_k$. Then, it is easy to know that $\widetilde{r}_{pq}^C = r_{pq}^d = 0$. By the conclusion $\widetilde{r}_{pq}^{B'} > \widetilde{r}_{pq}^C$ if $x_p, x_q \in U_1$, we have that $\widetilde{S}_{B'}(x_p) \supseteq \widetilde{S}_C(x_p)$. Therefore, $x_p \notin POS_{B'}^1(\{d\})$.

(3) $x_p \in U_2$, $x_q \in U_1$

Because of the symmetry between x_p and x_q, this case is the same as Case (2).

From the analyses of the three cases, we have that $\exists x_p$ such that $x_p \in POS_C^1(\{d\})$ but $x_p \notin POS_{B'}^1(\{d\})$, i.e. $POS_C^1(\{d\}) \neq POS_{B'}^1(\{d\})$.

In all, by Definition 4.5, we know that B is a Type-1 positive region reduct of C with respect to $\{d\}$.

($\Rightarrow$) Let B be a relative reduct of C. Then, we have $POS_B^1(\{d\}) = POS_C^1(\{d\})$. If $x_i \in POS_B^1(\{d\})$, $\exists Y_k \in U/\{d\}$ such that $\widetilde{S}_B(x_i) \subseteq Y_k$ and $x_i \in Y_k$. Because of $POS_B^1(\{d\}) = POS_C^1(\{d\})$, we have $x_i \in POS_C^1(\{d\})$ and $\widetilde{S}_C(x_i) \subseteq Y_k$. By Definition 4.5, we have $d(x_i) \neq d(x_j)$ if $m_{ij}^1 = \{a|\widetilde{r}_{ij}^a = \min_{b \in C}\{\widetilde{r}_{ij}^C\}\} \neq \emptyset$. Furthermore, by the definition of positive region, we have that for $\forall x_j \notin Y_k$, $\widetilde{r}_{ij}^B = \widetilde{r}_{ij}^C = 0$. Because of $B \subseteq C$, we have $m_{ij}^1 = \{a|\widetilde{r}_{ij}^a = \min_{b \in C}\{\widetilde{r}_{ij}^C\} = 0\} \supseteq \{a|\widetilde{r}_{ij}^a = \min_{b \in B}\{\widetilde{r}_{ij}^B\} = 0\}$. Thus, $m_{ij}^1 \cap B \neq \emptyset$.

If $x_i \notin POS_B^1(\{d\})$, then we have $\widetilde{S}_B(x_i) \subseteq Y_k$ for $\forall Y_k \in U/\{d\}$. Because of $POS_B^1(\{d\}) = POS_C^1(\{d\})$, $x_i \notin POS_C^1(\{d\})$, i.e. $x_i \notin U_1$. By Definition 4.5, if $m_{ij}^1 = \{a|\widetilde{r}_{ij}^a = \min_{b \in C}\{\widetilde{r}_{ij}^C\}\} \neq \emptyset$, x_j must be in U_1 and $d(x_i) \neq d(x_j)$, i.e. $x_j \in POS_C^1(\{d\})$ and $d(x_i) \neq d(x_j)$. Because of $POS_B^1(\{d\}) = POS_C^1(\{d\})$, $x_j \in POS_B^1(\{d\})$ and $d(x_i) \neq d(x_j)$. Hence, $\widetilde{r}_{ij}^B = \widetilde{r}_{ij}^C = 0$. Furthermore, it is easy to know $m_{ij}^1 = \{a|\widetilde{r}_{ij}^a = \min_{b \in C}\{\widetilde{r}_{ij}^C\} = \widetilde{r}_{ij}^C = 0\}$ and $B \subseteq C$, we have $m_{ij}^1 \cap B \neq \emptyset$.

By the existing condition that B is a Type-1 positive region reduct of C, we also know that for $\forall B' \subset B$, $POS_{B'}^1(\{d\}) \subseteq POS_C^1(\{d\})$. Therefore, $\exists x_p$ such that $x_p \in POS_C^1(\{d\})$ and $x_p \notin POS_{B'}^1(\{d\})$. Without any loss of generality, we suppose $x_p \in Y_k$. We have $\widetilde{r}_{pj}^C = 0$ for $\forall x_j \notin Y_k$ because of $x_p \in POS_C^1(\{d\})$, and $\exists x_q$ such that $\widetilde{r}_{pq}^{B'} > 0$ because of $x_p \notin POS_{B'}^1(\{d\})$. Hence, for $\forall a \in C$, if $\widetilde{r}_{pq}^a = 0$ then $a \notin B'$, and $m_{pq} = \{a|\widetilde{r}_{pq}^a = \min_{b \in C}\{\widetilde{r}_{pq}^C\} = 0\} \neq \emptyset$. Therefore, $B' \cap m_{pq} = \emptyset$. It follows that B is the minimal set that satisfies $B \cap m_{ij} \neq \emptyset$ for every $m_{ij} \neq \emptyset$. □

Sequently, based on the Type-1 discernibility matrix mentioned above, we define a new discernibility function: Type-1 discernibility function.

Definition 4.7. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. Type-1 discernibility function f_1 for SDT is a Boolean function of m Boolean variables $c_1^*, c_2^*, \ldots, c_m^*$ corresponding to the attribute $c_1, c_2, \ldots, c_m$, respectively, and defined as

$$f_1(c_1^*, c_2^*, \ldots, c_m^*) = \wedge\{\vee m_{ij}^1 \in M_{n\times n}^1, m_{ij}^1 \neq \emptyset\},$$

where $\vee m_{ij}^1$ is the disjunction of all variables c^* such that $a \in m_{ij}^1$ and $\wedge$ denotes conjunction.

The discernibility function $f_1(c_1^*, c_2^*, \ldots, c_m^*)$ describes the constraints which must hold to preserve discernibility in the sense of Tpye-1 positive region. For a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, the set of all prime implicants of $f_1(c_1^*, c_2^*, \ldots, c_m^*)$ determines the set of all Type-1 reducts of SDT.

Definition 4.8. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, then Type-1 core of C with respect to $\{d\}$ is defined as

$$Core_{\{d\}}^1(C) = \cap_{B \in \mathbf{RED}_{\{d\}}^1(C)} B,$$

where $\mathbf{RED}_{\{d\}}^1(C)$ is the set of all Type-1 positive region reducts.

The following theorem will give the approach to obtain Type-1 core of condition attribute set with respect to decision attribute set in a set-valued decision table.

Theorem 4.2. *Given a set-valued decision table* $SDT = (U, C \cup \{d\}, V, f)$, *then* $Core^1_{\{d\}}(C) = \{a : m^1_{ij} = \{a\}\}$.

Proof. $a \in Core^1_{\{d\}}(C) \Leftrightarrow POS^1_B(\{d\}) \neq POS^1_{B-\{a\}}(\{d\})$ for $\forall B \in \mathbf{RED}^1_{\{d\}}(C) \Leftrightarrow$ There exists $x_i \in POS^1_C(\{d\})$ such that $\widetilde{S}_{B-\{a\}}(x_i) \subset [x_i]_{\{d\}}$ ($[x_i]_{\{d\}} \in U/\{d\}$ *is a equivalent class that contains* x_i) $\Leftrightarrow$ There exists $x_i \in POS^1_C(\{d\})$ and $x_j \notin [x_i]_{\{d\}}$ such that $\widetilde{r}^{B-\{a\}}_{ij} > \widetilde{r}^B_{ij} = 0$ $\Leftrightarrow m^1_{ij} = \{c : \widetilde{r}^c_{ij} = \min_{b \in C}\{\widetilde{r}^b_{ij}\}, d(x_i) \neq d(x_j)$ and $x_i \in U_i\} = \{c : \widetilde{r}^c_{ij} = \min_{b \in C}\{\widetilde{r}^b_{ij}\}, d(x_i) \neq d(x_j)$ and $x_i \in POS^1_C(\{d\})\} = \{a\}$. □

The relationship between the elements of Dai's discernibility matrix and those of Type-1 discernibility matrix will be investigated in the following.

Theorem 4.3. *Given a set-valued decision table* $SDT = (U, C \cup \{d\}, V, f)$, m^{Dai}_{ij} *is an element in Dai's discernibility matrix of SDT, and* m^1_{ij} *is an element in Type-1 discernibility matrix of SDT, then*

$$m^1_{ij} \subseteq m^{Dai}_{ij},$$

especially, if $m^1_{ij} \neq \emptyset$ and $m^{Dai}_{ij} \neq \emptyset$ (*or* $m^1_{ij} = \emptyset$ and $m^{Dai}_{ij} = \emptyset$),

$$m^1_{ij} = m^{Dai}_{ij}.$$

Proof. For proving the theorem, there are four cases is analyzed as follows.

(1) $x_i \in U_1$

From the definition of m^1_{ij} and m^{Dai}_{ij}, we have that $m^1_{ij} = \{a \in C : \widetilde{r}^a_{ij} = \widetilde{r}^C_{ij}$ and $d(x_i) \neq d(x_j)\} = m^{Dai}_{ij}$, and it is impossible that m^1_{ij} and m^{Dai}_{ij} are empty sets.

(2) $x_i \in U_2$ and $x_j \in U_1$

In this case, $\widetilde{r}^{B'}_{ij} = \widetilde{r}^C_{ij} = 0$ if $d(x_i) \neq d(x_j)$. Hence, $m^1_{ij} = \{a \in C : \widetilde{r}^a_{ij} = \widetilde{r}^{B'}_{ij}$ and $d(x_i) \neq d(x_j)\} = \{c \in C : \widetilde{r}^a_{ij} = \widetilde{r}^C_{ij}$ and $d(x_i) \neq d(x_j)\} = m^{Dai}_{ij}$, and it is impossible that m^1_{ij} and m^{Dai}_{ij} are empty sets.

(3) $x_i, x_j \in U_2$ and $d(x_i) \neq d(x_j)$

In this case, $m^1_{ij} = \emptyset$, $m^{Dai}_{ij} = \{a \in C : \widetilde{r}^a_{ij} = \widetilde{r}^C_{ij}$ and $d(x_i) \neq d(x_j)\}$, then $m^{Dai}_{ij} \supset m^1_{ij}$, because it is impossible that m^{Dai}_{ij} is a empty set.

(4) $x_i, x_j \in U_2$ and $d(x_i) = d(x_j)$

In this case, $m^1_{ij} = \emptyset$, $m^{Dai}_{ij} = \emptyset$. Hence $m^1_{ij} = m^{Dai}_{ij}$. □

Corollary 4.1. *Given a set-valued decision table* $SDT = (U, C \cup \{d\}, V, f)$. *If* $Core^{Dai}_{\{d\}}(C)$ *is the Dai's core of C with respect to* $\{d\}$, *and* $Core^1_{\{d\}}(C)$ *is the Type-1 core of C with respect to* $\{d\}$, *then*

$$Core^1_{\{d\}}(C) \subseteq Core^{Dai}_{\{d\}}(C).$$

We omit the proof because the corollary is easily to be proved by means of Theorem 4.3.

To investigate the relationship between Dai's reducts and Type-1 reducts, we give the following theorem and remark.

Theorem 4.4. *Given a set-valued decision table* $SDT = (U, C \cup \{d\}, V, f)$, *if B* ($B \subseteq C$) *is a Dai's reduct of C with respect to* $\{d\}$, *then* $\exists$ *a Type-1 positive region reduct* B' *such that* $B \supseteq B'$.

Proof. From the exiting conditions, it is easy to know that $\widetilde{r}^B_{ij} = \widetilde{r}^C_{ij}$ *for* $x_i, x_j \in U \Rightarrow POS^1_C(\{d\}) = POS^1_B(\{d\})$ and $\widetilde{r}^{B-\{a\}}_{ij} \neq \widetilde{r}^C_{ij}$ *for* $x_i, x_j \in U \Rightarrow POS^1_C(\{d\}) \neq POS^1_{B-\{a\}}(\{d\})$ for $\forall a \in B$. Furthermore, by the definition of Type-1 positive region reducts, we have that there exists a Type-1 positive region reduct B' such that $B \supseteq B'$. □

From Theorem 4.4, we can see that, for each Dai's reduct, it is either a Type-1 positive region reduct or the superset of a Type-1 positive region.

Remark: By Theorem 4.3, without any loss of generality, we suppose that $m^{Dai}_{pq} = \{c_w\} \supset m^1_{pq} = \emptyset$, and $m^{Dai}_{ij} = m^1_{ij}$ for $\forall\ i \neq p, j \neq q$. The set of all Type-1 positive region reducts is $\mathbf{RED}^1_{\{d\}}(C) = \{red^1_{\{d\}}(C)_1, red^1_{\{d\}}(C)_2, \ldots, red^1_{\{d\}}(C)_{|\mathbf{RED}^1_{\{d\}}(C)|}\}$, $c_w \notin red^1_{\{d\}}(C)_i$, $1 \leq i \leq |\mathbf{RED}^1_{\{d\}}(C)|$. Thus, we have that

$$\begin{aligned} f_{Dai}(c^*_1, c^*_2, \ldots, c^*_m) &= \wedge\left\{\vee m^{Dai}_{ij} \in M^{Dai}_{n\times n}, m^{Dai}_{ij} \neq \emptyset\right\} \\ &= \vee_{k \leq |\mathbf{RED}^1_{\{d\}}(C)|}(c_w \wedge red^1_{\{d\}}(C)_k). \end{aligned}$$

By the expression of f_{Dai}, there exists $red^{Dai}_{\{d\}}(C)_u \in \mathbf{RED}^{Dai}_{\{d\}}(C)$ such that $c_w \cup red^1_{\{d\}}(C)_v \supseteq red^{Dai}_{\{d\}}(C)_u$. Three cases should be considered as follows.

(1) $c_w \cup red^1_{\{d\}}(C)_v = red^{Dai}_{\{d\}}(C)_u$, $c_w \notin red^1_{\{d\}}(C)_v$ and $c_w \in red^{Dai}_{\{d\}}(C)_u$. In this case, $red^1_{\{d\}}(C)_v \subset red^{Dai}_{\{d\}}(C)_u$;

(2) $c_w \cup red^1_{\{d\}}(C)_v = red^{Dai}_{\{d\}}(C)_u$, $c_w \in red^1_{\{d\}}(C)_v$ and $c_w \in red^{Dai}_{\{d\}}(C)_u$. In this case, $red^1_{\{d\}}(C)_v = red^{Dai}_{\{d\}}(C)_u$;

(3) $c_w \cup red^1_{\{d\}}(C)_v \supset red^{Dai}_{\{d\}}(C)_u$. By Theorem 4.3, the Type-1 positive region reducts in this case is the new reducts which have no explicit relationship with each of Dai's reducts.

From the above analyses, we can find out that among all Dai's reducts, some are exactly Type-1 positive region reducts, the others are the superset of the corresponding Type-1 positive region reducts. Additionally, some of Type-1 positive region reducts are the attribute set which are not explicitly relative with the Dai's reducts. The results indicate Type-1 positive region reducts have less redundancy and are more diverse (no lower number of reducts) than Dai's reducts.

In the sequence, based on the Type-2 positive region in Section 3, we can define Type-2 positive region reducts.

Definition 4.9. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, then B $(B \subseteq C)$ is Type-2 positive region reduct of C with respect to D if and only if

(1) $\forall x, y \in U, \widetilde{POS}_C^2(\{d\}) = \widetilde{POS}_B^2(\{d\})$;

(2) For any $B' \subset B, \widetilde{POS}_C^2(\{d\}) \neq \widetilde{POS}_{B'}^2(\{d\})$.

To obtain Type-2 positive region reducts, the following discernibility matrix is defined in the following.

Definition 4.10. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. Then Type-2 discernibility matrix is defined as $M_{n\times n}^2 = \{m_{ij}^2\}$, where

$$m_{ij}^2 = \begin{cases} \{a \in C : 1 - \widetilde{r}_{ij}^a \geq \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\}, & d(x_i) \neq d(x_j) \\ \emptyset, & d(x_i) = d(x_j) \end{cases}.$$

Furthermore, the following theorem is employed to demonstrate relationships between Type-2 positive region reducts and the elements in a Type-2 discernibility matrix, which is the theoretical foundation that assures all Type-2 positive region reducts can be obtained by a Type-2 discernibility matrix.

Theorem 4.5. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, then B $(B \subseteq C)$ is a Type-2 relative reduct of C if and only if B is the minimal set satisfying $B \cap m_{ij}^2 \neq \emptyset$ for $\forall m_{ij}^2 \neq \emptyset$.*

Proof. $(\Leftarrow)$ If $d(x_i) \neq d(x_j)$, then $m_{ij}^2 = \{a \in C : 1 - \widetilde{r}_{ij}^a \geq \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\}\}$. Let $B \cap m_{ij}^2 \neq \emptyset$ for $\forall m_{ij}^2 \neq \emptyset$ and $1 - \widetilde{r}_{ip}^C = \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\}$(i.e.$\widetilde{r}_{ip}^C = \max_{d(x_i)\neq d(x_k)}\{\widetilde{r}_{ik}^C\}$). For the development of the proof, we divide two cases as follows.

(1) For $x_j = x_p$

By the definition of discernibility matrix and the existing condition $B \cap m_{ij}^2 \neq \emptyset$ for $\forall m_{ij}^2 \neq \emptyset$, we have that $\exists c \in B$ such that $\widetilde{r}_{ip}^c \leq \widetilde{r}_{ip}^C = \min_{b\in C}\{\widetilde{r}_{ip}^b\}$. Because of $c \in C$, $\widetilde{r}_{ip}^c$ is not less than $\min_{b\in C}\{\widetilde{r}_{ip}^b\}$. We thus have that $\widetilde{r}_{ip}^c = \widetilde{r}_{ip}^C = \min_{b\in C}\{\widetilde{r}_{ip}^b\}$. Furthermore, because of $c \in B$ and $B \subseteq C$, we have that $\widetilde{r}_{ip}^C \leq \widetilde{r}_{ip}^B \leq \widetilde{r}_{ip}^c$, i.e. $\widetilde{r}_{ip}^B = \widetilde{r}_{ip}^C$.

(2) For $\forall x_j \neq x_p$

$\forall c \in B \cap m_{ij}^2$, $1 - \widetilde{r}_{ij}^c \geq \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\}$. Because of $c \in B$, we have that $\widetilde{r}_{ij}^B \leq \widetilde{r}_{ij}^c \leq \widetilde{r}_{ip}^C$.

In all, from the conclusion of the two cases, we have that $\widetilde{r}_{ip}^B = \widetilde{r}_{ip}^C$ and $\widetilde{r}_{ij}^B \leq \widetilde{r}_{ip}^C$ for $x_j \neq x_p$. Hence, $\max_{d(x_i)\neq d(x_k)}\{\widetilde{r}_{ik}^B\} = \widetilde{r}_{ip}^C = \max_{d(x_i)\neq d(x_k)}\{\widetilde{r}_{ik}^C\}$. Then, it is easy to obtain $\mu_{\underline{\widetilde{C}}(Y)}(x_i) = \min_{x_j\notin Y}\{1 - \widetilde{r}_{ij}^C\} = 1 - \max_{x_j\notin Y}\{\widetilde{r}_{ij}^C\} = 1 - \max_{x_j\notin Y}\{\widetilde{r}_{ij}^B\} = \min_{x_j\notin Y}\{1 - \widetilde{r}_{ij}^B\} = \mu_{\underline{\widetilde{B}}(Y)}(x_i)$.

$(\Rightarrow)$ For any $m_{ij}^2 \neq \emptyset$, we have $d(x_i) \neq d(x_j)$ by definition.

Let B be a relative reduct of C. Then, we have that $\mu_{\underline{\widetilde{C}}(Y)}(x_i) = \min_{d(x_i)\neq d(x_j)}\{1 - \widetilde{r}_{ij}^C\} = \min_{d(x_i)\neq d(x_j)}\{1 - \widetilde{r}_{ij}^B\} = \mu_{\underline{\widetilde{B}}(Y)}(x_i)$, i.e. $\max_{d(x_i)\neq d(x_j)}\{\widetilde{r}_{ij}^C\} = \max_{d(x_i)\neq d(x_j)}\{\widetilde{r}_{ij}^B\}$, $\widetilde{r}_{ij}^B \leq \max_{d(x_i)\neq d(x_j)}\{\widetilde{r}_{ij}^C\}$, and for $\forall x_i$, $x_j \in U$ and $d(x_i) \neq d(x_j)$, $\exists c \in B$, such that $\widetilde{r}_{ij}^c = \min_{b\in B}\{\widetilde{r}_{ij}^b\} = \widetilde{r}_{ij}^B$. Thus, $\exists c \in B$, $\widetilde{r}_{ij}^c \leq \max_{d(x_i)\neq d(x_k)}\{\widetilde{r}_{ik}^C\}$, i.e.$1 - \widetilde{r}_{ij}^c \geq \min_{d(x_i)\neq d(x_j)}\{1 - \widetilde{r}_{ij}^C\}$. In summary, we have $c \in m_{ij}^2$ and $B \cap m_{ij}^2 \neq \emptyset$.

If $a \in B'$, we have that $\widetilde{r}_{ip}^a = \min_{b\in B'}\{\widetilde{r}_{ip}^b\} = \widetilde{r}_{ip}^{B'} = \widetilde{r}_{ip}^C$ because of $B' \subseteq C$. Then $\min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^{B'}\} \geq \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\} = 1 - \widetilde{r}_{ip}^{B'}$, and $\min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^{B'}\} = 1 - \widetilde{r}_{ip}^{B'}$. Therefore, $\min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^{B'}\} = \min_{d(x_i)\neq d(x_k)}\{1 - \widetilde{r}_{ik}^C\}$, which is contract with B' is not a reduct. Therefore, $a \notin B$, i.e. $B \cap m_{ij}^2 = \emptyset$. □

Sequently, based on Type-2 discernibility matrix we proposed, we define a new discernibility function: Type-2 discernibility function.

Definition 4.11. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. Type-2 discernibility function f_2 for SDT is a Boolean function of m Boolean variables $c_1^*, c_2^*, \ldots, c_m^*$ corresponding to the attributes $c_1, c_2, \ldots, c_m$, respectively, and defined as

$$f_2(c_1^*, c_2^*, \ldots, c_m^*) = \wedge\{\vee m_{ij}^2 : m_{ij}^2 \in M_{n\times n}^2, m_{ij}^2 \neq \emptyset\},$$

where $\bigvee m_{ij}^2$ is the disjunction of all variables c^* such that $a \in m_{ij}^2$ and $\bigwedge m_{ij}^2$ is the conjunction of them.

Definition 4.12. Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. Type-2 core of C with respect to $\{d\}$ is defined as

$$Core_{\{d\}}^2(C) = \cap_{B\in \mathbf{RED}_{\{d\}}^2(C)} B,$$

where $\mathbf{RED}^2_{\{d\}}(C)$ is the set of all Type-2 positive region reducts.

The following theorem will give the approach to obtain Type-2 core of condition attribute set with respect to decision attribute set in a set-valued decision table.

Theorem 4.6. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, then $Core^2_{\{d\}}(C) = \{a : m^2_{ij} = \{a\}\}$.*

Proof. $a \in Core^2_{\{d\}}(C) \Leftrightarrow POS^2_C(\{d\}) \neq POS^2_{C-\{a\}}(\{d\})$ for $\forall B \in \mathbf{RED}^2_{\{d\}}(C) \Leftrightarrow$ There exists $x_i \in U$ such that $\max_{Y \in U/\{d\}}\{\min_{x_j \notin Y}\{1 - \widetilde{r}^{C-\{a\}}_{ij}\}\} \neq \max_{Y \in U/\{d\}}\{\min_{x_j \notin Y}\{1 - \widetilde{r}^{C}_{ij}\}\} \Leftrightarrow$ There exists $x_i \in U$ such that $\min_{x_j \notin [x_i]_{\{d\}}}\{1 - \widetilde{r}^{C-\{a\}}_{ij}\} \neq \min_{x_j \notin [x_i]_{\{d\}}}\{1 - \widetilde{r}^{C}_{ij}\} \Leftrightarrow m^2_{ij} = \{c : \widetilde{r}^{c}_{ij} \leq 1 - \min_{x_k \notin [x_i]_{\{d\}}}\{1 - \widetilde{r}^{C}_{ik}\}, d(x_i) \neq d(x_j)$ and $x_i \in U_i\} = \{a\}$. □

Theorem 4.6 states that a attribute belongs to Type-2 core if it is the sole attribute contained in an element in a Type-2 discernibility matrix.

The relationships between the elements of a Dai's discernibility matrix and those of a Type-2 discernibility matrix will be investigated in the following theorem.

Theorem 4.7. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. m^{Dai}_{ij} is an element in Dai's discernibility matrix of SDT, and m^2_{ij} is an element in Type-2 discernibility matrix of SDT, then*

$$m^2_{ij} \supseteq m^{Dai}_{ij},$$

especially, when $\widetilde{r}^{C}_{ij} = \widetilde{r}^{max}_{ij}$, or $\widetilde{r}^{C}_{ij} = \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}$, or $m^2_{ij} = \emptyset$ and $m^{Dai}_{ij} = \emptyset$,

$$m^2_{ij} = m^{Dai}_{ij}.$$

where $\widetilde{r}^{max}_{ij} = \max_{a \in C}\{\widetilde{r}^{a}_{ij}\}$.

Proof. For the development of the proof, four possible cases are analyzed as follows.

(1) $\widetilde{r}^{C}_{ij} \leq \widetilde{r}^{max}_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}$ and $d(x_i) \neq d(x_j)$

In this case, $m^2_{ij} = \{a \in C : \widetilde{r}^{a}_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}\} = C \supseteq \{a \in C : \widetilde{r}^{a}_{ij} = \min_{a \in C}\{\widetilde{r}^{a}_{ij}\}\} = m^{Dai}_{ij}$. Especially, if $\widetilde{r}^{C}_{ij} = \widetilde{r}^{max}_{ij}$, $m^2_{ij} = m^{Dai}_{ij}$.

(2) $\widetilde{r}^{C}_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\} < \widetilde{r}^{max}_{ij}$ and $d(x_i) \neq d(x_j)$

In this case, $m^2_{ij} = \{a \in C : \widetilde{r}^{a}_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}\} \supseteq \{a \in C : \widetilde{r}^{a}_{ij} = \min_{a \in C}\{\widetilde{r}^{a}_{ij}\}\} = m^{Dai}_{ij}$. Especially, if $\widetilde{r}^{C}_{ij} = \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}$, $m^2_{ij} = m^{Dai}_{ij}$.

(3) $\max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\} < \widetilde{r}^{C}_{ij} \leq \widetilde{r}^{max}_{ij}$ and $d(x_i) \neq d(x_j)$

It is obviously impossible that $\widetilde{r}^{C}_{ij} > \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^{C}_{ik}\}$. Hence, this case cannot appear.

(4) $d(x_i) = d(x_j)$

In this case, $m^2_{ij} = \emptyset$, $m^{Dai}_{ij} = \emptyset$. Hence $m^2_{ij} = m^{Dai}_{ij}$. □

Corollary 4.2. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. If $Core^{Dai}_{\{d\}}(C)$ is a Dai's core of C with respect to $\{d\}$, and $Core^2_{\{d\}}(C)$ is the core of C with respect to $\{d\}$ in the sense of Type-2 positive region, then*

$$Core^2_{\{d\}}(C) \subseteq Core^{Dai}_{\{d\}}(C).$$

We omit the proof because it is easy to prove the corollary by means of Theorem 4.7.

To investigate the relationship between Dai's reducts and Type-2 reducts, we give the following theorem and remark.

Theorem 4.8. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. If B ($B \subseteq C$) is a Dai's reduct, then $\exists$ a Type-2 positive region reduct B' such that $B \supseteq B'$.*

The proof of this theorem is similar with that of Theorem 4.4, hence we omit it.

From Theorem 4.8, we can see that, for each Dai's reduct, it is either a Type-2 positive region reduct or the superset of a Type-2 positive region. **Remark:** By Theorem 4.7, without any loss of generality, we suppose that $m^2_{pq} = \{c_w\} \cup m^{Dai}_{pq}$, $c_w \notin m^{Dai}_{pq}$, and $m^{Dai}_{ij} = m^2_{ij}$ for $\forall\ i \neq p,\ j \neq q$. Thus, $\widetilde{r}^{c_w}_{pq} > \widetilde{r}^{C}_{pq}$. The set of all positive region reducts is $\mathbf{RED}^2_{\{d\}}(C) = \{red^2_{\{d\}}(C)_1, red^2_{\{d\}}(C)_2, \ldots, red^2_{\{d\}}(C)_{|\mathbf{RED}^2_{\{d\}}(C)|}\}$, $c_w \notin red^2_{\{d\}}(C)_i$, $1 \leq i \leq |\mathbf{RED}^2_{\{d\}}(C)|$. Thus, we have that

$$\begin{aligned} f(M^2) &= \wedge\left\{\vee m^2_{ij}, m^2_{ij} \neq \emptyset\right\} \\ &= \left(\wedge\left\{\vee_{i \neq p, j \neq q} m^2_{ij}, m^2_{ij} \neq \emptyset\right\}\right) \wedge (\vee m^2_{pq}) \\ &= \left(\wedge\left\{\vee_{i \neq p, j \neq q} m^2_{ij}, m^2_{ij} \neq \emptyset\right\}\right) \wedge (\vee(\{c_w\} \cup m^{Dai}_{pq})). \end{aligned}$$

Furthermore, without any loss of generality, we assume that $(\wedge\{\vee_{i \neq p, j \neq q} m^{Dai}_{ij}, m^{Dai}_{ij} \neq \emptyset\}) = red'_{\{d\}}(C)_1 \vee red'_{\{d\}}(C)_2 \vee \cdots \vee red'_{\{d\}}(C)_l$. Three cases will be considered as follows.

(1) $c_w \in red'_{\{d\}}(C)_k$. In this case, we have that $\vee\{red'_{\{d\}}(C)_k \wedge \{\vee(m_{pq}^{Dai} \cup \{c_w\})\}\} = red'_{\{d\}}(C)_k$ and $red'_{\{d\}}(C)_k \in \mathbf{RED}^2_{\{d\}}(C)$. For any conjunctive term A in the formula obtained by reducing $\vee\{red'_{\{d\}}(C)_k \wedge \{\vee m_{pq}^{Dai}\}\}$ to the simplest form, it is easy to know $A \in \mathbf{RED}^{Dai}_{\{d\}}(C)$ and $red'_{\{d\}}(C)_k \subset A$.

(2) $c_w \notin red'_{\{d\}}(C)_k$ and $red'_{\{d\}}(C)_k \cap m_{pq}^{Dai} \neq \emptyset$. In this case, we have that $\vee\{red'_{\{d\}}(C)_k \wedge \{\vee(m_{pq}^{Dai} \cup \{c_w\})\}\} = red'_{\{d\}}(C)_k$. Hence $red'_{\{d\}}(C)_k \in \mathbf{RED}^2_{\{d\}}(C)$. And we have that $\vee\{red'_{\{d\}}(C)_k \wedge \{\vee(m_{pq}^{Dai})\}\} = red'_{\{d\}}(C)_k$. Hence $red'_{\{d\}}(C)_k \in \mathbf{RED}^{Dai}_{\{d\}}(C)$.

(3) $c_w \notin red'_{\{d\}}(C)_k$ and $red'_{\{d\}}(C)_k \cap m_{pq}^{Dai} = \emptyset$. In this case, we have that for $red'_{\{d\}}(C)_k \cup \{c_w\} \in \mathbf{RED}^2_{\{d\}}(C)$ and $red'_{\{d\}}(C)_k \cup \{c_w\} \notin \mathbf{RED}^{Dai}_{\{d\}}(C)$. For any conjunctive term A in the formula obtained by reducing $\vee\{red'_{\{d\}}(C)_k \wedge \{\vee m_{pq}^{Dai}\}\}$ to the simplest form, it is easy to know that $A \in \mathbf{RED}^{Dai}_{\{d\}}(C)$ and $red'_{\{d\}}(C)_k \cup \{c_w\} \cup \{a\} \supset A$ for $\forall a \in m_{pq}^{Dai}$. In other words, the union of the Type-2 reducts of some attribute is superset of Dai's reducts in this case. By Theorem 4.7, Type-2 positive region reducts in this case is the new reducts which have no explicit relationship with Dai's reducts.

From the above analyses, we can find out that among all Dai's reducts, some of them are exactly Type-2 positive region reducts, the others are the superset of the corresponding Type-2 positive region reducts. Additionally, some of Type-2 positive region reducts are the attribute sets which are not explicitly relative with Dai's reducts. The results indicate Type-2 positive region reducts have less redundancy and are more diverse (no lower number of reducts) than Dai's reducts.

Sequently, we will investigate the relationships between Type-1 positive region reducts and Type-2 positive region reducts.

Theorem 4.9. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, if $B \subseteq C$ is a Type-2 positive region reduct, then $\exists$ a Type-1 positive region reduct B' such that $B \supseteq B'$.*

Proof. From the exiting conditions, it is easy to know $\widetilde{POS}^2_C(\{d\}) = \widetilde{POS}^2_B(\{d\}) \Rightarrow \max_{Y \in U/\{d\}}\{\min_{x_j \notin Y}\{1 - \widetilde{r}^B_{ij}\}\} = \max_{Y \in U/\{d\}}\{\min_{x_j \notin Y}\{1 - \widetilde{r}^C_{ij}\}\} \Rightarrow \min_{x_j \notin Y}\{1 - \widetilde{r}^B_{ij}\} = \min_{x_j \notin Y}\{1 - \widetilde{r}^C_{ij}\}$ and $x_i \in Y \Rightarrow$ if $\widetilde{r}^C_{ij} = 0$ for $\forall x_j \notin Y$ then $\widetilde{r}^B_{ij} = 0$ for $\forall x_j \notin Y$ and if $\exists x_j \notin Y$ such that $\widetilde{r}^C_{ij} > 0$ then $\widetilde{r}^B_{ij} \geq \widetilde{r}^C_{ij} > 0 \Rightarrow POS^1_C(\{d\}) = POS^1_B(\{d\})$. In the similar way, $\widetilde{POS}^2_C(\{d\}) \neq \widetilde{POS}^2_{B-\{a\}}(\{d\})$ for $\forall a \in B \Rightarrow POS^1_C(\{d\}) \neq POS^1_{B-\{a\}}(\{d\})$ for $\forall a \in B$. Furthermore, by the definition of Type-1 positive region reducts, we have that there exists a Type-1 positive region reduct B' such that $B \supseteq B'$. □

The relationship between the elements of a Type-1 discernibility matrix and those of a Type-2 discernibility matrix will be demonstrated in the following.

Theorem 4.10. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. m_{ij}^{Dai} is an element in Dai's discernibility matrix of SDT, and m^1_{ij} is an element in Type-1 discernibility matrix of SDT, then*

$$m^2_{ij} \supseteq m^1_{ij},$$

especially, if $m^2_{ij} \neq \emptyset$ and $m^1_{ij} \neq \emptyset$ *(or* $m^2_{ij} = \emptyset$ and $m^1_{ij} = \emptyset$*),*

$$m^2_{ij} = m^1_{ij}.$$

Proof. We first suppose $\widetilde{r}^{max}_{ij} = \max_{a \in C}\{\widetilde{r}^a_{ij}\}$. For the development of the proof, four possible cases are analyzed as follows.

(1) $\widetilde{r}^C_{ij} \leq \widetilde{r}^{max}_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\}$ and $d(x_i) \neq d(x_j)$

If $x_i \in U_1$ and $d(x_i) \neq d(x_k)$, then $\widetilde{r}^C_{ik} = 0$ for $\forall x_k \in U$. Hence $m^2_{ij} = C = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} = 0\} = \{a \in C : \widetilde{r}^a_{ij} = \min_{a \in C}\{\widetilde{r}^a_{ij}\} = 0\} = m^1_{ij}$. If $x_i \in U_2$, $x_j \in U_1$ and $d(x_j) \neq d(x_i)$, then $\widetilde{r}^C_{ij} = 0$ for $\forall x_j \in U$. Hence $m^2_{ij} = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} = 0\} = C = \{a \in C : \widetilde{r}^a_{ij} = \min_{a \in C}\{\widetilde{r}^a_{ij}\} = 0\} = m^1_{ij}$. If $x_i \in U_2$, $x_j \in U_2$ and $d(x_j) \neq d(x_i)$, then $m^1_{ij} = \emptyset$. Hence $m^2_{ij} = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\}\} = C \supset m^1_{ij}$.

(2) $\widetilde{r}^C_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} < \widetilde{r}^{max}_{ij}$ and $d(x_i) \neq d(x_j)$

If $x_i \in U_1$ and $d(x_i) \neq d(x_k)$, then $\widetilde{r}^C_{ik} = 0$ for $\forall x_k \in U$. Hence $m^2_{ij} = C = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} = 0\} = \{a \in C : \widetilde{r}^a_{ij} = \min_{a \in C}\{\widetilde{r}^a_{ij}\} = 0\} = m^1_{ij}$. If $x_i \in U_2$, $x_j \in U_1$ and $d(x_j) \neq d(x_i)$, then $\widetilde{r}^C_{ij} = 0$ for $\forall x_j \in U$. Hence $m^2_{ij} = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} = 0\} = C = \{a \in C : \widetilde{r}^a_{ij} = \min_{a \in C}\{\widetilde{r}^a_{ij}\} = 0\} = m^1_{ij}$. If $x_i \in U_2$, $x_j \in U_2$ and $d(x_j) \neq d(x_i)$, then $m^1_{ij} = \emptyset$. Hence $m^2_{ij} = \{a \in C : \widetilde{r}^a_{ij} \leq \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\}\} \supset m^1_{ij}$.

(3) $\max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\} < \widetilde{r}^C_{ij} \leq \widetilde{r}^{max}_{ij}$ and $d(x_i) \neq d(x_j)$

It is obviously impossible that $\widetilde{r}^C_{ij} > \max_{d(x_i) \neq d(x_k)}\{\widetilde{r}^C_{ik}\}$.

(4) $d(x_i) = d(x_j)$

In this case, $m^2_{ij} = \emptyset$, $m^{Dai}_{ij} = \emptyset$. Hence $m^2_{ij} = m^1_{ij}$. □

Corollary 4.3. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$. If $Core^1_{\{d\}}(C)$ is Type-1 core of C with respect to $\{d\}$, and $Core^2_{\{d\}}(C)$ is the core of C with respect to $\{d\}$ in the sense of Type-2 positive region, then*

$$Core^1_{\{d\}}(C) \subseteq Core^2_{\{d\}}(C).$$

Table 1
Data set: Soybean(Large).

U	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\cdots$	d
x_1	6	0	2	1	0	1	1	1	0	0	$\cdots$	1
x_2	4	0	2	1	0	2	0	2	1	1	$\cdots$	1
x_3	3	0	2	1	0	1	0	2	1	2	$\cdots$	1
x_4	3	0	2	1	0	1	0	2	0	1	$\cdots$	1
x_5	6	0	2	1	0	2	0	1	0	2	$\cdots$	1
x_6	5	0	2	1	0	3	0	1	0	1	$\cdots$	1
x_7	5	0	2	1	0	2	0	1	1	0	$\cdots$	1
x_8	4	0	2	1	1	1	0	1	0	2	$\cdots$	1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{301}	3	$*$	$*$	$*$	$*$	2	1	$*$	$*$	$*$	$\cdots$	17
x_{302}	4	$*$	$*$	$*$	$*$	2	1	$*$	$*$	$*$	$\cdots$	17
x_{303}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$	18
x_{304}	1	1	$*$	0	$*$	1	0	$*$	$*$	$*$	$\cdots$	19
x_{305}	0	1	$*$	0	$*$	0	3	$*$	$*$	$*$	$\cdots$	19
x_{306}	1	1	$*$	0	$*$	0	0	$*$	$*$	$*$	$\cdots$	19
x_{307}	1	1	$*$	0	$*$	1	3	$*$	$*$	$*$	$\cdots$	19

We omit the proof because it is easy to prove the corollary by means of Theorem 4.10.

By Corollary 4.1-4.3, it is easily to know $Core^1_{\{d\}}(C) \subseteq Core^2_{\{d\}}(C) \subseteq Core^{Dai}_{\{d\}}(C)$.

To investigate the relationship between Type-1 positive region reducts and Type-2 positive region reducts, we give the following theorem and remark.

Theorem 4.11. *Given a set-valued decision table $SDT = (U, C \cup \{d\}, V, f)$, if $B \subseteq C$ is a Type-2 reduct, then $\exists$ a Type-1 positive region reduct B' such that $B \supseteq B'$.*

The proof of the theorem is similar with the one of Theorem 4.4, hence we omit it.

From Theorem 4.11, we can see that for each Type-2 reduct, it is either a Type-1 positive region reduct or the superset of a Type-1 positive region.

Remark: The relationship between Type-1 positive region reducts and Type-2 positive region reducts is similar with the one between Type-1 positive region reducts and Type-2 positive region reducts. Therefore, without the concrete analysis, we only list three cases of their relationship as follows.

(1) A Type-1 positive region reduct is the real subset of some Type-2 positive region reduct.

(2) A Type-1 positive region reduct is equal to some Type-2 positive region reduct.

(3) A Type-1 positive region reduct is the attribute set which has no explicit relationship with all Type-2 positive region reducts.

From the above analyses, we can find, among all Type-2 reducts, some of them are exactly Type-1 positive region reducts, the others are the superset of some Type-1 positive region reducts. Additionally, some of Type-1 positive region reducts are the attribute sets which are not explicitly relative with all Type-2 positive region reducts. The results indicate Type-1 positive region reducts have less redundancy and are more diverse (no lower number of reducts) than Type-2 reducts.

5. Experimental analysis

In this section, we use the same data set Soybean(Large) [4] and Soybean(large)-test [42] to show the effectiveness and performance of the two new types of fuzzy rough approximations for set-valued data. Data set Soybean(Large) (Shown in Table 1) and Soybean(large)-test contain 307 objects and 376 objects, respectively. And, in the two data sets, there are 35 conditional attributes ($a_1, a_2, \ldots, a_{35}$), one decision attribute (d), and some attribute values are missed. All the attributes are categorical attributes, and missing values are denoted by $*$ in the two data sets.

To facilitate our experiment, we convert the two data sets into a set-valued decision table by replacing missing values $*$ on an attribute with the set of containing all values of the attribute, as shown in Table 2. For example, a_2 is the second conditional attribute “plant-stand” in the original data set, whose domain $\{normal, lt - normal\}$ is indicated by {0, 1}: 0 indicates “normal” and 1 indicates “lt-normal”. If we employ the set value {0, 1} to represent a missing value on this attribute, the attribute “plant-stand” becomes a set-valued attribute. If we do the same change to every attribute in Soybean(Large), the data set will become a set-valued decision table.

5.1. Experiments on effectiveness of reducts

We will illustrate the effectiveness of two proposed fuzzy rough approximations by comparing Dai’s reducts with the reducts obtained by the new rough approximations. To conduct the experiments, we first construct Dai’s, Type-1 and Type-2 discernibility matrices on data set Soybean(Large) (shown inTable 3–5). For the limit of the length of the paper, we only give some elements of these discernibility matrices.

Table 2
Set-valued decision table converted from Soybean(Large).

U	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	$\cdots$	d
x_1	6	0	2	1	0	1	1	1	0	0	$\cdots$	1
x_2	4	0	2	1	0	2	0	2	1	1	$\cdots$	1
x_3	3	0	2	1	0	1	0	2	1	2	$\cdots$	1
x_4	3	0	2	1	0	1	0	2	0	1	$\cdots$	1
x_5	6	0	2	1	0	2	0	1	0	2	$\cdots$	1
x_6	5	0	2	1	0	3	0	1	0	1	$\cdots$	1
x_7	5	0	2	1	0	2	0	1	1	0	$\cdots$	1
x_8	4	0	2	1	1	1	0	1	0	2	$\cdots$	1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{301}	3	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	2	1	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	17
x_{302}	4	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	2	1	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	17
x_{303}	{0, 1, 2, 3, 4, 5, 6}	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	{0, 1, 2, 3}	{0, 1, 2, 3}	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	18
x_{304}	1	1	{0, 1, 2}	0	{0, 1}	1	0	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{305}	0	1	{0, 1, 2}	0	{0, 1}	0	3	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{306}	1	1	{0, 1, 2}	0	{0, 1}	0	0	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{307}	1	1	{0, 1, 2}	0	{0, 1}	1	3	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19

Table 3
Dai's discernibility matrix on data set Soybean(Large).

D	x_1	x_2	$\cdots$	x_{33}	$\cdots$	x_{303}	$\cdots$	x_{307}
x_1	$\emptyset$	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_6, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_2	–	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_6, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_3	–	–	$\cdots$	$\{a_1, a_2, a_4, a_6, a_7, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{307}	–	–	$\cdots$	–	$\cdots$	–	$\cdots$	$\emptyset$

Table 4
Type-1 discernibility matrix on data set Soybean(Large).

D	x_1	x_2	$\cdots$	x_{33}	$\cdots$	x_{303}	$\cdots$	x_{307}
x_1	$\emptyset$	$\emptyset$	$\cdots$	$\emptyset$	$\cdots$	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_2	–	$\emptyset$	$\cdots$	$\emptyset$	$\cdots$	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_6, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_3	–	–	$\cdots$	$\emptyset$	$\cdots$	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{307}	–	–	$\cdots$	–	$\cdots$	–	$\cdots$	$\emptyset$

Table 5
Type-2 discernibility matrix on data set Soybean(Large).

D	x_1	x_2	$\cdots$	x_{33}	$\cdots$	x_{303}	$\cdots$	x_{307}
x_1	$\emptyset$	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_6, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_1, a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_2	–	$\emptyset$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_1, a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_6, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
x_3	–	–	$\cdots$	$\{a_1, a_2, a_4, a_6, a_7, a_{22}, a_{24}, a_{35}\}$	$\cdots$	$\{a_1, a_{17}\}$	$\cdots$	$\{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{307}	–	–	$\cdots$	–	$\cdots$	–	$\cdots$	$\emptyset$

From Table 3 and 4, we can see that an element in the Dai's discernibility matrix derived from Table 2 is equal to that in the corresponding element of the Type-1 discernibility matrix derived from Table 2 if both of these two elements are not empty sets, such as $m^{Dai}_{1,307} = \{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\} = m^2_{1,307}$, and an element in the Type-1 discernibility matrix must be an empty set if it is a proper subset of the element in the Dai's discernibility matrix, such as $m^{Dai}_{1,33} = \{a_1, a_2, a_4, a_6, a_{22}, a_{24}, a_{35}\} \supset m^1_{1,33} = \emptyset$. The experimental results are consistent with the theoretical results in Section 4.

From Table 3 and Table 5, we can see that an element in the Dai's discernibility matrix derived from Table 2 is the subset of the corresponding element of the Type-2 discernibility matrix derived from Table 2 if both of these two elements are not empty sets, such as $m^{Dai}_{1,307} = \{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\} = m^2_{1,307}$ and $m^{Dai}_{1,33} = \{a_{17}\} \subset m^2_{1,33} = \{a_1, a_{17}\}$. The experimental results are consistent with the theoretical conclusion in Section 4.

From Table 4 and 5, we can see that an element in the Type-1 discernibility matrix derived from Table 2 is equal to that in the corresponding element of the Type-2 discernibility matrix derived from Table 2 if both of these two elements are not empty sets, such as $m^1_{1,307} = \{a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}\} = m^2_{1,307}$, and an element of the Type-1 discernibility

Table 6
Comparison of Dai's reducts and Type-1 positive region reducts on data set Soybean(Large).

No.	$Red_D^{Dai}(C)$	*No.*	$Red_D^1(C)$	*Relation*
Red_1^{Dai}	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_1^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_1^{Dai} = red_1^1$
Red_2^{Dai}	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_2^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_2^{Dai} = red_2^1$
…	…	…	…	…
Red_{3490}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	Red_{3490}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	$red_{3490}^{Dai} = red_{3490}^1$
–	–	Red_{3491}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_9, a_{12}, a_{15}, a_{16}, a_{17}, a_{23}, a_{28}, a_{30}, a_{35}$	–
–	–	Red_{3492}^1	$a_1, a_2, a_3, a_4, a_6, a_7, a_9, a_{12}, a_{15}, a_{16}, a_{17}, a_{20}, a_{23}, a_{28}, a_{30}, a_{35}$	–
…	…	…	…	
–	–	Red_{4557}^1	$a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}, a_{12}, a_{14}, a_{16}, a_{17}, a_{19}, a_{27}, a_{31}, a_{32}, a_{33}, a_{35}$	–

Table 7
Comparison of Dai's reducts and Type-2 positive region reducts on data set Soybean(Large).

No.	$Red_D^{Dai}(C)$	*No.*	$Red_D^1(C)$	*Relation*
Red_1^{Dai}	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_1^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_1^{Dai} = red_1^1$
Red_2^{Dai}	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_2^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_2^{Dai} = red_2^1$
…	…	…	…	…
Red_{3490}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	Red_{3490}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	$red_{3490}^{Dai} = red_{3490}^1$

Table 8
Comparison of Type-2 positive region reducts and Type-1 positive region reducts on data set Soybean(Large).

No.	$Red_D^2(C)$	*No.*	$Red_D^1(C)$	*Relation*
Red_1^2	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_1^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_1^{Dai} = red_1^1$
Red_2^2	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	Red_2^1	$a_1, a_2, a_4, a_7, a_{13}, a_{14}, a_{15}, a_{17}, a_{28}, a_{35}$	$red_2^{Dai} = red_2^1$
…	…	…	…	…
Red_{3490}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	Red_{3490}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{24}, a_{35}$	$red_{3490}^{Dai} = red_{3490}^1$
–	–	Red_{3491}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_9, a_{12}, a_{15}, a_{16}, a_{17}, a_{23}, a_{28}, a_{30}, a_{35}$	–
–	–	Red_{3492}^1	$a_1, a_2, a_3, a_4, a_6, a_7, a_9, a_{12}, a_{15}, a_{16}, a_{17}, a_{20}, a_{23}, a_{28}, a_{30}, a_{35}$	–
…	…	…	…	
–	–	Red_{4557}^1	$a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{10}, a_{12}, a_{14}, a_{16}, a_{17}, a_{19}, a_{27}, a_{31}, a_{32}, a_{33}, a_{35}$	–

matrix must be an empty set if it is a proper subset of the element in the Type-2 discernibility matrix, such as $m_{1,33}^2 = \{a_1, a_2, a_4, a_6, a_{22}, a_{24}, a_{35}\} \supset m_{1,33}^1 = \emptyset$. The experimental results are consistent with the theoretical conclusion in Section 4.

By the discernibility matrices shown in Table 3-5, we can obtain all of the Dai's reducts, the Type-1 positive region reducts and the Type-2 positive region reducts from Table 2, and Table 6–8 show the comparison of these reducts. From Table 6, we can see that the number of the Dai's reducts is 3490, the number of the Type-1 positive region reducts is 4557. Among these Type-1 positive region reducts, there are 3490 reducts that are identical with some reduct in these Dai's reducts, and 1067 Type-1 positive region reducts are the reducts that cannot be found by the Dai's discernibility matrix. From Table 7, we can see that the number of Type-2 positive region reducts derived from Table 2 is 3490, and they are all some reduct in the Dai's reducts (the Type-2 positive region reducts derived from Table 2 are all identical with those Dai's reducts, respectively). Thus, the relationship between Type-2 positive region reducts and Type-1 positive region reducts is the same as that between the Dai's reducts and the Type-1 positive region reducts derived from Table 2, which is demonstrated by Table 8.

The experimental results on data set Soybean(Large) cannot thoroughly illustrate the differences among Dai's reducts, Type-1 positive region reducts, and Type-2 positive region reducts derived from a decision table. In order to better exhibit the differences, we construct a new set-valued decision table by deleting the attributes in $Core_D^{Dai}(C)$ of Soybean(Large) in which the condition attribute set becomes $\{a_2, a_3, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{11}, a_{13}, a_{14}, a_{15}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}\}$ (shown in Table 9), and we will investigate the differences among Dai's reducts, Type-1 positive region reducts, and Type-2 positive region reducts derived from the modified data set.

From Table 10, we can see that the number of Dai's reducts is 265 and the number of Tpye-1 positive region reducts is 271, which are derived from Table 9. Among the 265 Dai's reducts, there are 30 reducts in which each reduct is identi-

Table 9
Modified Soybean(Large) by deleting five attributes.

U	a_2	a_3	a_4	a_5	a_6	a_8	a_9	a_{10}	$\cdots$	d
x_1	0	2	1	0	1	1	0	0	$\cdots$	1
x_2	0	2	1	0	2	2	1	1	$\cdots$	1
x_3	0	2	1	0	1	2	1	2	$\cdots$	1
x_4	0	2	1	0	1	2	0	1	$\cdots$	1
x_5	0	2	1	0	2	1	0	2	$\cdots$	1
x_6	0	2	1	0	3	1	0	1	$\cdots$	1
x_7	0	2	1	0	2	1	1	0	$\cdots$	1
x_8	0	2	1	1	1	1	0	2	$\cdots$	1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{301}	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	2	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	17
x_{302}	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	2	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	17
x_{303}	{0, 1}	{0, 1, 2}	{0, 1, 2}	{0, 1}	{0, 1, 2, 3}	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	18
x_{304}	1	{0, 1, 2}	0	{0, 1}	1	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{305}	1	{0, 1, 2}	0	{0, 1}	0	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{306}	1	{0, 1, 2}	0	{0, 1}	0	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19
x_{307}	1	{0, 1, 2}	0	{0, 1}	1	{0, 1, 2}	{0, 1, 2}	{0, 1, 2}	$\cdots$	19

Table 10
Comparison of Dai's reducts and Type-1 positive region reducts on modified Soybean(Large).

No.	$Red_D^{Dai}(C)$	No.	$Red_D^1(C)$	Relationship
Red_1^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{27}, a_{30}$	Red_1^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{27}, a_{30}$	$Red_1^{Dai} = Red_1^1$
Red_2^{Dai}	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{13}, a_{17}, a_{23}, a_{30}$	Red_2^1	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{13}, a_{17}, a_{23}, a_{30}$	$Red_2^{Dai} = Red_2^1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Red_{30}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{29}, a_{30}$	Red_{30}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{29}, a_{30}$	$Red_{30}^{Dai} = Red_{30}^1$
Red_{31}^{Dai}	$a_1, a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{28}, a_{30}$	Red_{31}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{28}, a_{30}$	$Red_{31}^{Dai} \supset Red_{31}^1$
Red_{32}^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{16}, a_{17}, a_{28}, a_{30}$	Red_{31}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{28}, a_{30}$	$Red_{32}^{Dai} \supset Red_{31}^1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Red_{35}^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{28}, a_{30}$	Red_{31}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{28}, a_{30}$	$Red_{35}^{Dai} \supset Red_{31}^1$
Red_{35}^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{28}, a_{30}$	Red_{32}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{28}$	$Red_{35}^{Dai} \supset Red_{32}^1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Red_{265}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{19}, a_{24}, a_{28}, a_{30}$	Red_{149}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{24}, a_{28}, a_{30}$	$Red_{265}^{Dai} \supset Red_{149}^1$
–	–	Red_{150}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{14}, a_{17}, a_{26}, a_{30}$	–
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
–	–	Red_{271}^1	$a_1, a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{27}, a_{29}, a_{30}$	–

cal with some reduct in the Type-1 positive region reducts, and the others are the supersets of the Type-1 positive region reducts. It should be point out that among 271 Tpye-1 positive region reducts, there are 122 reducts have no explicit relationship with each of the Dai's reducts. It can be said that the attribute reduction method based a Type-1 discernibility matrix can obtain 122 new reducts. The experimental results are consistent with the theoretical results in Section 4.

From Table 11, we can see that the number of Dai's reducts is 265 and the number of Type-2 positive region reducts is 382, which are derived from Table 9. Among the 265 Dai's reducts, there are 187 reducts in which each reduct is identical with some reduct in the Type-2 positive region reducts, and the others are the supersets of the Type-2 positive region reducts. It should be point out that 136 Type-2 positive region reducts have no explicit relationship with each of these Dai's reducts (i.e. the attribute reduction method based Type-2 discernibility matrix can find 136 new reducts). The experimental results are consistent with the theoretical results in Section 4.

From Table 12, we can see that the number of Type-1 reducts is 271 and the number of Tpye-2 positive region reducts is 382, which are derived from Table 9. Among the 382 Type-2 reducts, there are 38 reducts in which each reduct is identical with some reduct in the Type-1 positive region reducts, and the others are the supersets of the Type-2 positive region reducts. It should be point out that 59 Type-1 positive region reducts have no explicit relation with each of Type-2 reducts. It can be said that the attribute reduction method based Type-1 discernibility matrix can find 59 reducts that cannot be found by the attribute reduction method based Type-2 discernibility matrix. The experimental results are consistent with the theoretical results in Section 4.

Furthermore, we illustrate the relationships among the Dai's core, Type-1 core and Type-2 core through computing these types of cores on the set-valued decision tables shown in Table 2 and 9, which are exhibited in Table 13. From Table 13, we can find these types of cores and the relationships among them as follows:

$Core_{\{d\}}^{Dai}(C) = \{a_1, a_7, a_{12}, a_{16}, a_{17}\}$, $Core_{\{d\}}^{1}(C) = \{a_7, a_{12}, a_{16}, a_{17}\}$, and $Core_{\{d\}}^{2}(C) = \{a_1, a_7, a_{12}, a_{16}, a_{17}\}$.

It is obvious that $Core_{\{d\}}^{Dai}(C) \supseteq Core_{\{d\}}^{2}(C) \supseteq Core_{\{d\}}^{1}(C)$.

Table 11
Comparison of Dai's reducts and Type-2 positive region reducts on modified Soybean(Large).

No.	$Red_D^{Dai}(C)$	No.	$Red_D^2(C)$	Relationship
Red_1^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{27}, a_{30}$	Red_1^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{27}, a_{30}$	$Red_1^{Dai} = Red_1^2$
Red_2^{Dai}	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{13}, a_{17}, a_{23}, a_{30}$	Red_2^2	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{12}, a_{13}, a_{17}, a_{23}, a_{30}$	$Red_2^{Dai} = Red_2^2$
...	...	...	...	...
Red_{187}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{23}, a_{29}, a_{30}$	Red_{187}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{23}, a_{29}, a_{30}$	$Red_{187}^{Dai} = Red_{187}^2$
				...
Red_{188}^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{28}, a_{30}$	Red_{188}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{28}$	$Red_{188}^{Dai} \supset Red_{188}^2$
Red_{189}^{Dai}	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{17}, a_{23}, a_{30}$	Red_{189}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{17}, a_{23}$	$Red_{189}^{Dai} \supset Red_{189}^2$
...	...	...	...	...
Red_{262}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{23}, a_{28}, a_{30}$	Red_{245}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{28}, a_{30}$	$Red_{262}^{Dai} \supset Red_{245}^2$
Red_{263}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{19}, a_{21}, a_{28}, a_{30}$	Red_{244}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{21}, a_{28}, a_{30}$	$Red_{263}^{Dai} \supset Red_{244}^2$
Red_{264}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{19}, a_{22}, a_{28}, a_{30}$	Red_{245}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{28}, a_{30}$	$Red_{264}^{Dai} \supset Red_{245}^2$
Red_{265}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{19}, a_{24}, a_{28}, a_{30}$	Red_{243}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{19}, a_{24}, a_{28}, a_{30}$	$Red_{265}^{Dai} \supset Red_{243}^2$
Red_{265}^{Dai}	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{19}, a_{24}, a_{28}, a_{30}$	Red_{246}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{24}, a_{28}, a_{30}$	$Red_{265}^{Dai} \supset Red_{246}^2$
–	–	Red_{247}^2	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}$	–
...	...	...	...	...
–	–	Red_{382}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{14}, a_{19}, a_{22}, a_{23}, a_{29}, a_{30}$	–

Table 12
Comparison of Type-1 reducts and Type-2 positive region reducts on modified Soybean(Large).

No.	$Red_D^2(C)$	No.	$Red_D^1(C)$	Relationship
Red_1^2	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}$	Red_1^1	$a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}$	$Red_1^2 = Red_1^1$
Red_2^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{17}, a_{23}$	Red_2^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{11}, a_{12}, a_{13}, a_{17}, a_{23}$	$Red_2^2 = Red_2^1$
...	...	...	...	...
Red_{38}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{29}, a_{30}$	Red_{38}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{29}, a_{30}$	$Red_{38}^2 = Red_{38}^1$
Red_{39}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{25}$	Red_{39}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{17}, a_{23}, a_{25}$	$Red_{39}^2 \supset Red_{39}^1$
Red_{40}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{13}, a_{17}, a_{23}, a_{26}$	Red_{40}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{17}, a_{23}, a_{26}$	$Red_{40}^2 \supset Red_{40}^1$
Red_{41}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{14}, a_{17}, a_{23}, a_{25}$	Red_{39}^1	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{17}, a_{23}, a_{25}$	$Red_{41}^2 \supset Red_{39}^1$
Red_{42}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{14}, a_{17}, a_{23}, a_{26}$	Red_{32}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{17}, a_{23}, a_{26}$	$Red_{42}^2 \supset Red_{40}^1$
Red_{42}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{14}, a_{17}, a_{23}, a_{26}$	Red_{32}^2	$a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{12}, a_{14}, a_{23}, a_{26}$	$Red_{42}^2 \supset Red_{41}^1$
...	...	...	...	...
Red_{382}^2	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{19}, a_{22}, a_{23}, a_{29}, a_{30}$	Red_{212}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_{10}, a_{12}, a_{16}, a_{22}, a_{23}, a_{29}, a_{30}$	$Red_{382}^2 \supset Red_{212}^1$
–	–	Red_{213}^1	$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{23}, a_{24}$	–
...	...	...	...	...
–	–	Red_{271}^1	$a_1, a_2, a_3, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{12}, a_{16}, a_{19}, a_{27}, a_{29}, a_{30}$	–

Table 13
Three types of cores in Soybean(Large) and Modified Soybean(Large).

No.	Soybean(Large)	Modified Soybean(Large)
$Core_D^{Dai}(C)$	$a_1, a_7, a_{12}, a_{16}, a_{17}$	$a_2, a_3, a_5, a_6, a_7, a_8, a_{12}, a_{30}$
$Core_D^1(C)$	$a_7, a_{12}, a_{16}, a_{17}$	$a_2, a_3, a_5, a_6, a_7, a_8, a_{12}$
$Core_D^2(C)$	$a_1, a_7, a_{12}, a_{16}, a_{17}$	$a_2, a_3, a_5, a_6, a_7, a_8, a_{12}$

In summary, all the experimental results given in this section well verify the theoretical results in Section 4.

5.2. Experiments on performance of reducts

In this subsection, we conduct some experiments to demonstrate the advantages of the new reducts found by our proposed rough approximation. Since reducts are essential to define rules, it is necessary to evaluate the reducts by employing *coverage degree* and *percentage of correct classification* which are two important assessment indexes of decision rules.

We will use the method of extracting rules in Ref. [12]. The rules extracted from a set-valued decision table ($SDT = (U, C)$) can be formulated by $t \to s$, where $t = \wedge(c, \tau)$, $c \in B \subseteq C$, $\tau \in V_c - \{*\}$, and $s = (d, w)$, $w \in V_d$. Sequently, we will call t and s condition and decision part of a rule, respectively.

Table 14
Comparison between Dai's and Type-1 reducts from the perspective of coverage degree.

		$Acov(R_{D=T1})$	$Acov(R_{D\supset T1})$	$Acov(R_{T1\subset D})$	$Acov(R_{T1-D})$	$Acov(R_D)$	$Acov(R_{T1})$
S	Dai's Method	0.0700	N/A	N/A	N/A	0.0700	N/A
	Type-1Method	0.0700	N/A	N/A	0.0770	N/A	0.0715
MS	Dai's Method	0.0746	0.0739	N/A	N/A	0.0739	N/A
	Type-1Method	0.0746	N/A	0.0745	0.0748	N/A	0.0747

Note: where *S* represents *Soybean*(*large*), and *MS* represents *Modified Soybean*(*large*).

Table 15
Comparison between Dai's and Type-2 reducts from the perspective of coverage degree.

		$Acov(R_{D=T2})$	$Acov(R_{D\supset T2})$	$Acov(R_{T2\subset D})$	$Acov(R_{T2-D})$	$Acov(R_D)$	$Acov(R_{T2})$
S	Dai's Method	0.0700	N/A	N/A	N/A	0.0700	N/A
	Type-2 Method	0.0700	N/A	N/A	N/A	N/A	0.0700
MS	Dai's Method	0.0741	0.0734	N/A	N/A	0.0739	N/A
	Type-2 Method	0.0741	N/A	0.0742	0.0746	N/A	0.0743

Note: where *S* represents *Soybean*(*large*), and *MS* represents *Modified Soybean*(*large*).

Table 16
Comparison between Type-1 and Type-2 reducts from the perspective of coverage degree.

		$Acov(R_{T1=T2})$	$Acov(R_{T1\subset T2})$	$Acov(R_{T2\supset T1})$	$Acov(R_{T1-T2})$	$Acov(R_{T1})$	$Acov(R_{T2})$
S	Tpye-1Method	0.0700	N/A	N/A	0.0770	0.0715	N/A
	Type-2Method	0.0700	N/A	N/A	N/A	N/A	0.0700
SM	Tpye-1Method	0.0751	N/A	0.0747	0.0743	0.0747	N/A
	Type-2Method	0.0751	0.0742	N/A	N/A	N/A	0.0743

Note: where *S* represents *Soybean*(*large*), and *MS* represents *Modified Soybean*(*large*).

To compare the performance of reducts from the perspective of *coverage degree* of rules, we first extract a set of rules from a reduced incomplete decision table by means of the above definition of rules, and then the coverage degree can be denoted by the following formular [13,24,31]:

$$cov(t \to s) = \frac{card(\|t\| \cap \|s\|)}{card\|s\|},$$

where $\|t\| = \{x \in U : v \in a(x), \forall(a, v) \in t\}$ and $\|s\| = \{x \in U : w \in d(x), (d, w) \in s\}$. Furthermore, we introduce *Average coverage degree* (*Acov*) of a set of rules extracted from a set-valued decision table *SDT* as follows:

$$Acov(RuleSet_{SDT}) = \frac{\sum_{t\to s\in RuleSet_{SDT}} cov(t \to s)}{|RuleSet_{SDT}|},$$

where $RuleSet_{SDT}$ indicates the set of all rules extracted from a set-valued decision table *SDT*, and $|RuleSet_{SDT}|$ is the number of rules in $RuleSet_{SDT}$.

To detect the difference between the sets of all reducts obtained by Dai's, Type-1 and Type-2 discernibility matrices from the perspective of *coverage degree*, we need another criterion to evaluate the set of reducts *RedSet*, which is given as follows:

$$RAcov(RedSet) = \frac{\sum_{SDT\in SDT_{RedSet}} Acov(RuleSet_{SDT})}{|RedSet|},$$

where *RedSet* is a set of reducts, and SDT_{RedSet} indicates the set of all the data sets derived from *RedSet*.

Furthermore, to facilitate the display of experimental results, based on the conclusions in Section 4, we can denote the set of all the Type-1 reducts as R_{T1}, and $R_{T1} = R_{T1=D} \cup R_{T1\subset D} \cup R_{T1-D}$, where $R_{T1=D}$ is the set of the reducts which are Dai's reducts and Type-1 reducts simultaneously, and $R_{T1\subset D}$ indicates a set of Type-1 reducts, each of which is the subset of some Dai's reduct, R_{T1-D} is the set of Type-1 reducts, each of which is neither a Dai's reduct nor a subset of some Dai's reduct. And the set of all the Dai's reducts can be denoted as R_D, and $R_D = R_{T1=D} \cup R_{D\supset T1}$, where $R_{D\supset T1}$ is a subset of R_D in which each reduct includes at least one Type-1 reduct. It is needs to be emphasized that the reducts in $R_{T1\subset D}$ and R_{T1-D} are The newfound reducts by our proposed Type-1 discernibility matrix. We will employ Table 14 to shows the advantage of these new reducts from the perspective of *coverage degree*. From Table 14, we can see that for Soybean(large), $RAcov(R_{T1-D}) = 0.0770 > RAcov(R_D) = 0.0700$, which illustrates our newfound reducts have better performance in the sense of *coverage degree*. In a similar way, Table 15 and 16 display the comparisons between Dai's and Type-2 reducts, and between Type-1 and Type-2 reducts, respectively. From the Table 15, we can draw the same conclusion that is consistent with that from Table 14, i.e. the newfound Type-2 reducts work better than those derived by Dai's methods. Table 16 indicates that the Type-1 reducts are more optimal than Type-2 ones.

Table 17
Comparison between Dai's and Type-1 reducts from the perspective of percentage of correct classification.

		$Apcc(R_{D=T1})$	$Apcc(R_{D\supset T1})$	$Apcc(R_{T1\subset D})$	$Apcc(R_{T1-D})$	$Apcc(R_D)$	$Apcc(R_{T1})$
S-St	Dai's Method	66.55%	N/A	N/A	N/A	66.55%	N/A
	Type-1 Method	66.55%	N/A	N/A	72.77%	N/A	68.00%
MS-MSt	Dai's Method	52.53%	50.37%	N/A	N/A	50.61%	N/A
	Type-1 Method	52.53%	N/A	51.81%	53.05%	N/A	52.45%

Note: where $S-St$ represents that $Soybean(large)$ and $Soybean(large)-test$ are regarded as training set and testing set respectively, and $MS-MSt$ represents $Modified\,Soybean(large)$ and $Modified\,Soybean(large)-test$ are regarded as training set and testing set respectively.

Table 18
Comparison between Dai's and Type-2 reducts from the perspective of percentage of correct classification.

		$Apcc(R_{D=T2})$	$Apcc(R_{D\supset T2})$	$Apcc(R_{T2\subset D})$	$Apcc(R_{T2-D})$	$Apcc(R_D)$	$Apcc(R_{T2})$
S-St	Dai's Method	66.55%	N/A	N/A	N/A	66.55%	N/A
	Type-2 Method	66.55%	N/A	N/A	N/A	N/A	66.55%
MS-MSt	Dai's Method	50.95%	49.81%	N/A	N/A	50.61%	N/A
	Type-2 Method	50.95%	N/A	49.99%	51.09%	N/A	50.85%

Note: where $S-St$ represents that $Soybean(large)$ and $Soybean(large)-test$ are regarded as training set and testing set respectively, and $MS-MSt$ represents $Modified\,Soybean(large)$ and $Modified\,Soybean(large)-test$ are regarded as training set and testing set respectively.

Sequently, to evaluate the performance of reducts more comprehensively, we introduce *percentage of correct classification* of classifier (pcc), which can be denoted as:

$$pcc(Data_{Tra}, Data_{Tes}) = \frac{\sum_{x_i\in U'} diff(d(x_i) - d_{reduct}(x_i))}{|U'|},$$

where $Data_{Tra} = (U, C)$ is a training data set, $Data_{Tes} = (U', C)$ is a testing data set, U' is the objects in $Data_{Tes}$, $x_i \in U'$, $d(x_i)$ is the original label, $d'(x_i)$ is the label given by classifier constructed based on $Data_{Tra}$, and $diff(a, b) = 0$ if $a = b$, otherwise $diff(a, b) = 1$.

To calculate $pcc(Data_{Tra}, Data_{Tes})$, we need to construct a classifier based on rules, in which how to match a new object with a rule is a key issue. We thus introduce a type of distance to solve the problem as follows:

$$dis_C(x_i, r) = \sum_{c\in C} dis_c(x_i, r),$$

where $d_c(x_i, r) = 1 - \frac{|a(x_i)\cap a(t_r)|}{|a(x_i)\cup a(t_r)|}$, t_r indicates the condition part of Rule r, and $a(t_r)$ is the value of t_r on Attribute a.

Furthermore, by means of the distance, we design a classifier based on rules, which can be described as follows:

Algorithm 1: An classifier based on rules.
Input: A training set-valued data set $Data_{Tra} = (U, C)$ and a testing set-valued data set $Data_{Tes}(U', C)$;
Output: $pcc(Data_{Tra}, Data_{Tes})$.
Step 1: Extract a set of rules R from the training data set;
Step 2: For each object $x_i \in U'$
{For each rule $r \in R$,
{Calculate the distance $d(x_i, r)$;}
Find a rule $r^* = arg\min_{r\in R} dis(x_i, r)$;
$d'(x_i) := w(r*)$;}
Step 3: Calculate $pcc(Data_{Tra}, Data_{Tes})$;
Step 4: End,
where, $w(r*)$ indicates the decision value w of the condition part of $r*$.

Furthermore, we present *Average pcc* ($Apcc$) to assess the performance of a set of reducts as follows:

$$Apcc(RedSet) = \frac{\sum_{reduct\in RedSet} pcc(Data_{Tra}^{reduct}, Data_{Tes}^{reduct})}{|RedSet|},$$

where $Data_{Tra}^{reduct}$ indicates a training data set in which *reduct* is its attribute set, and $Data_{Tes}^{reduct}$ indicates a testing data set in which *reduct* is its attribute set.

Sequently, we conduct experiments on the training data sets (Soybean(large) and Modified Soybean(large)) and the testing data set (Soybean(large)-test and Modified Soybean(large)-test that is constructed by the same method to construct Modified Soybean(large)). Table 17, 18 and 19 are used to illustrate the advantage of the newfound reducts by our proposed methods from the perspective of *percentage of correct classification*. Table 17 shows that when Soybean(large) and Soybean(large)-test are regarded as the training data set and the testing data set respectively, the $Apcc(R_{T1-D}) = 0.7277 > Apcc(R_D) = 0.6655$, which indicates the newfound reducts is better than the old ones in the sense of *percentage of correct classification*. Similarly, Table 18 and 19 display the comparison between Dai's and Type-2 reducts, and the

Table 19
Comparison between Type-1 and Type-2 reducts from the perspective of percentage of correct classification.

		$Apcc(R_{T1=T2})$	$Apcc(R_{T1\subset T2})$	$Apcc(R_{T2\supset T1})$	$Apcc(R_{T1-T2})$	$Apcc(R_{T1})$	$Apcc(R_{T2})$
S-St	Tpye-1 reducts	66.55%	N/A	N/A	72.77%	68.00%	N/A
	Type-2 reducts	66.55%	N/A	N/A	N/A	N/A	66.55%
MS-MSt	Tpye-1 reducts	52.55%	N/A	52.31%	52.81%	52.45%	N/A
	Type-2 reducts	52.55%	50.66%	N/A	N/A	N/A	50.85%

Note: where $S-St$ represents that $Soybean(large)$ and $Soybean(large)-test$ are regarded as training set and testing set respectively, and $MS-MSt$ represents $Modified\,Soybean(large)$ and $Modified\,Soybean(large)-test$ are regarded as training set and testing set respectively.

comparison of between Type-1 and Type-2 reducts, respectively. From Table 18, the similar results, which is that Type-2 reducts are better than Dai's reducts, can still be got. And the experimental results in Table 19 illustrate the Type-1 reducts are more optimal than Type-2 reducts from the perspective of *percentage of correct classification*.

In all, from the experimental results, it is evident that the newfound reducts obtained by our proposed methods are more optimal than those by Dai's method.

6. Conclusions

In this paper, we have proposed two new types of lower and upper approximations and their corresponding positive regions for set-valued data, based on which two new types of discernibility matrices and discernibility functions were constructed. These new discernibility matrixes and discernibility functions enables us to acquire some new reducts that can not be found by means of Dai's method. We introduced some theorems to theoretically demonstrate that these newfound reducts have less redundancy and are more diverse (no lower number of reducts) than the Dai's reducts. Experiments have been conducted on some data sets from UCI to verify the theoretical results and to illustrate the newfound reducts are more optimal than Dai's reducts from the respective of coverage degree of rules and percentage of correct classification.

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