Set-based granular computing: A lattice model

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A R T I C L E   I N F O

A B S T R A C T

Set-based granular computing plays an important role in human reasoning and problem solving. Its three key issues constitute information granulation, information granularity and granular operation. To address these issues, several methods have been developed in the literature, but no unified framework has been formulated for them, which could be inefficient to some extent. To facilitate further research on the topic, through consistently representing granular structures induced by information granulation, we introduce a concept of knowledge distance to differentiate any two granular structures. Based on the knowledge distance, we propose a unified framework for set-based granular computing, which is named a lattice model. Its application leads to desired answers to two key questions: (1) what is the essence of information granularity, and (2) how to perform granular operation. Through using the knowledge distance, a new axiomatic definition to information granularity, called generalized information granularity is developed and its corresponding lattice model is established, which reveal the essence of information granularity in set-based granular computing. Moreover, four operators are defined on granular structures, under which the algebraic structure of granular structures forms a complementary lattice. These operators can effectively accomplish composition, decomposition and transformation of granular structures. These results show that the knowledge distance and the lattice model are powerful mechanisms for studying set-based granular computing.

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1. Introduction

Granular computing (GrC) originally proposed by Zadeh [54] plays a fundamental role in human reasoning and problem solving (see [48]). Its three basic issues are information granulation, organization and causation. As pointed out in [53–57], the information granulation involves decomposition of whole into parts; the organization involves integration of parts into whole; and the causation involves association of causes with effects. They have been applied in relevant fields such as interval analysis, cluster analysis, machine learning, databases, data mining and knowledge discovery. The research on granular computing has led to four important methods, which are rough set theory [8,10–12,25,26,42,60], fuzzy set theory [13,14,24], concept lattice theory [22,23,40,45], and quotient space theory [58].

Set-based granular computing is a type of granular computing whose concerned universe is characterized by a finite set. It is a dominant research task in granular computing. The following concepts are elementary for set-based granular

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computing. *A granule* is a clump of objects or points drawn together by indistinguishability, similarity, and proximity of functionality [57]. Granulation of the universe leads to a collection of granules. *A granular structure* is a mathematical structure of the collection of granules, in which the inner structure of each granule is visible (a granule is a white box) and the interactions among granules are detected by the visible structures. *A knowledge base* is a collection of some granular structures on the same universe. Given these concepts, an important task is to establish a conceptual framework for granular computing. For our further development, we will briefly review several focuses of attention in granular computing. They include a measure of granularity [15–17,31,36,43,46], information processing [21], a framework of granular computing [18,20,33], problem solving based on "granulate and conquer" principle and quotient theory [60], dynamic granulation [32,35], multigranulation view [34,37], granular computing for human-centric information processing [1–3,52], and their applications [27–29,59]. Yao et al. [49] have reviewed perspectives and challenges in granular computing in the first ten years of its development. It can be seen from the developments that granular computing has been evolving into a field of cross-disciplinary study. In set-based granular computing, there are three key issues to address, which are information granulation, information granularity and granular operation.

Information granulation is to construct a set of information granules by a given granulation strategy in set-based granular computing. It is a pretreatment step of granular computing and the basis of problem solving with it. Information granulation will adopt various strategies according to user's acquirements and targets of problem solving. Common granulation strategies are mainly based on binary relations [8,10–12,16,21], clustering analysis [4], proximity of functionality [6] and image segmentation [7], and so on. In a broad sense, each of granulation strategies can be seen as the information granulation induced by a specific binary relation. For example, image segmentation is to partition an image to a set of sub-images in which each sub-image can be seen as an equivalence class, and hence this strategy can be characterized by an equivalent relation. The granulation of objects induced by an equivalent relation is a set of equivalence classes, in which each equivalence class can be regarded as a (Pawlak) information granule [25,26]; the granulation of objects induced by a tolerance relation generates a set of tolerance classes, in which each tolerance class also can be seen as a tolerance information granule [16]. By using a neighborhood relation, objects are granulated into a set of information granules, and each neighborhood is called a neighborhood information granule [19,44]. Dick et al. [9] proposed a granular neural network, and researched its performance analysis and application to re-granulation. However, these granular structures have no unified knowledge representation, which could be inefficient to some extent for establishing a conceptual framework for set-based granular computing.

Information granularity in set-based granular computing is an index to measure the granulation degree of objects in a given data set. How to calculate the information granularity of a granular structure has always been a key problem. In general, the information granularity represents discernibility ability of information in a granular structure. The smaller the information granularity, the stronger its discernibility ability. To date, several forms of information granularity have been proposed according to various views and targets [15,17,31,43,46]. Wierman [43] introduced the concept of granulation measure to quantify the uncertainty of information in a knowledge base. This concept has the same form as Shannon's entropy under the axiom definition. Liang et al. [15,16] proposed information granularity in either of complete and incomplete data sets, which have been effectively applied in attribute significance measure, feature selection, rule extraction, etc. Qian and Liang [31] presented combination granulation with an intuitive knowledge-content nature to measure the size of information granulation in a knowledge base. Xu et al. [46] gave an improved measure for roughness of a rough set in rough set theory proposed by Pawlak [25], which is also an information granularity in a broad sense. In the above forms of information granularity, the partial order relation plays a key role in characterizing the monotonicity of each of them. Although these excellent research contributions have been made in the context of set-based granular computing, there remains an important issue to be addressed. What is the essence of measuring an information granularity? As mentioned by Zadeh, in general, information granularity should characterize the granulation degree of objects from the viewpoint of hierarchy [54]. This provides a point of view that an information granularity should characterize hierarchical relationships among granular structures. To answer the question, in this investigation, we will develop an axiomatic approach to information granularity in set-based granular computing.

Granular operation is to answer the problem how to achieve composition, decomposition and transformation of information granules/granular structures. This problem is also one of key tasks in set-based granular computing [18–21,33,44,47,50,51]. Granular operation has two types of operations. One is operation of information granules, and the other is operation of granular structures. Lin [18–21] proposed a kind of operators among information granules, called knowledge operation, in which each information granule is a neighborhood. Yao [50,51] gave another approach to operation of information granules in a neighborhood system. Wu et al. [44] investigate how to perform operation of information granules in k-step-neighborhood systems. Yang et al. [47] modified Lin's version for much better operation of information granules. If we see operation of information granules in the context of granular structures, these operators can be used to generate new granular structures, and hence operation of information granules can be seen as inner operation of granular structures [33]. To perform operation of granular structures, Qian et al. [33] proposed four operators on tolerance granular structures. This is an attempt to study composition, decomposition and transformation of granular structures in set-based granular computing. In this study, we focus on the second granular operation, i.e., operation of granular structures with the unified knowledge representation.

From the research progresses above, it can be seen that many important results have been developed in the literature, however, there is no unified framework for these developments, which could be inefficient to some extent for studying

set-based granular computing. This is the driving force of our research. In this paper, we intend to solve two key problems: (1) what is the essence of information granularity, and (2) how to perform granular operation. We will address these two problems from geometric view and algebraic view, respectively. By consistently representing granular structures induced by information granulation, we first introduce a concept of knowledge distance to differentiate any two granular structures. Based on the knowledge distance, we then propose a unified framework of set-based granular computing in terms of geometric view, named a lattice model. In the proposed lattice model, through introducing several partial order relations, a series of axiomatic definitions to information granularity are developed, which reveal the essence of information granularity in set-based granular computing. Four operators are presented in granular structures, under which the algebraic structure of granular structures on the lattice model is a complementary lattice. These operators can achieve composition, decomposition and transformation of granular structures.

The rest of this paper is organized as follows. Some basic concepts in set-based granular computing are briefly reviewed, and a unified knowledge representation of granular structures is given in Section 2. In Section 3, we introduce a concept of knowledge distance to differentiate any two granular structures, and investigate geometry properties on granular structures based on the knowledge distance. In Section 4, through revealing the limitations of two existing axiomatic definitions of information granularity, we develop a new axiomatic approach to information granularity induced by the knowledge distance, called a generalized information granularity. Through these results, we also establish a lattice model to the generalized information granularity, which solves the problem of what is the essence of information granularity in set-based granular computing. In Section 5, we introduce four operators to achieve composition, decomposition and transformation of granular structures, under which the algebraic structure of granular structures is a complementary distributive lattice, which provides a strategy for performing granular operation in set-based granular computing. Finally, Section 6 concludes this paper with some remarks and discussions.

2. Knowledge representation in set-based granular computing

In this section, we review several basic concepts, and give a unified knowledge representation of granular structures for the present study.

In rough set theory, as defined by Pawlak, a knowledge base is denoted by $(U, \mathbf{R}) = (U, R_1, R_2, \ldots, R_m)$, where R_i is an equivalence relation [8,10–12,30,38]. U/R constitutes a partition of U, called a granular structure on U, and every equivalence class is called a Pawlak information granule [36]. In a broad sense, an information granularity denotes the average measure of Pawlak information granules (equivalence classes) induced by R.

If R_i $(i = 1, 2, \ldots, m)$ is a tolerance relation satisfying reflexivity and symmetry, then $(U, \mathbf{R}) = (U, R_1, R_2, \ldots, R_m)$ can be called a tolerance knowledge base [16]. Let $SIMR$ denote the family of sets $\{S_R(x), \ x \in U\}$, the granular structure induced by R, where $S_R(x) = \{y \mid xRy\}$, R is a tolerance relation. A member $S_R(x)$ from $SIMR$ will be called a tolerance information granule. In fact, $\{S_R(x): \ x \in U\}$ is a binary neighborhood system (BNS). For a tolerance knowledge base, in a broad sense, an information granularity denotes average measure of tolerance information granules (tolerance classes) induced by R.

If R_i $(i = 1, 2, \ldots, m)$ is a neighborhood relation, then $(U, \mathbf{R}) = (U, R_1, R_2, \ldots, R_m)$ can be called a neighborhood knowledge base [17]. A neighborhood relation R on the universe is a relation matrix $M(R) = (r_{ij})_{n \times n}$, where

$$r_{ij} = \begin{cases} 1, & d(x_i, x_j) \leqslant \varepsilon, \\ 0, & \text{otherwise}, \end{cases}$$

where d is a distance [13,14] between x and y, ε is a non-negative number. Let $A, B \subseteq AT$ be categorical and numerical attributes, respectively. The neighborhood granules of objects x induced by A, B, $A \cup B$ are defined as

(1) $N_A(x) = \{x_i \in U \mid d_A(x, x_i) = 0\}$;
(2) $N_B(x) = \{x_i \in U \mid d_B(x, x_i) \leqslant \varepsilon\}$;
(3) $N_{A \cup B}(x) = \{x_i \in U \mid d_A(x, x_i) = 0 \wedge d_B(x, x_i) \leqslant \varepsilon\}$.

Let $N(U)$ denote the family of sets $\{N_R(x), \ x \in U\}$, the granular structure induced by R. A member $N_R(x)$ from $N(U)$ will be called a neighborhood. For a neighborhood knowledge base, in a broad sense, an information granularity denotes the average measure of neighborhood information granules (tolerance classes) induced by R.

The above granular structures play a fundamental role in set-based granular computing, however, there is no unified knowledge representation for them, which could be inconvenient to establish a conceptual framework for set-based granular computing.

For a general binary relation satisfying reflexivity, one can uniformly represent the granular structure induced by $P \in \mathbf{R}$ as a vector $K(P) = (N_P(x), \ x \in U)$, where $N_P(x)$ is the neighborhood (or information granule) induced by an object $x \in U$ with respect to P. In fact, $\{N_P(x), \ x \in U\}$ is a binary neighborhood system (BNS). For a given granular structure $K(P)$, one can define its complement granular structure $\wr K(P)$ as follows:

$$\wr K(P) = \left\{ \wr N_P(x) \,\middle|\, \wr N_P(x) = x \cup \sim N_P(x), x \in U \right\}, \tag{1}$$

where $\sim N_P(x) = U - N_P(x)$.

In particular, let $K(\omega) = (N_P(x) \mid N_P(x) = \{x\}, \ x \in U)$ denote the finest granular structure, and $K(\omega) = (N_P(x) \mid N_P(x) = U, \ x \in U)$ the roughest granular structure in this study. In fact, we have that $K(\omega) = \wr K(\delta)$ and $K(\delta) = \wr K(\omega)$.

The above modes of information granulation in which the granules are crisp (c-granular) play an important role in a wide variety of methods in set-based granular computing.

Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$. We define a partial order relation $\preccurlyeq$ as follows: $K(P) \preccurlyeq K(Q)$ $(P, Q \in \mathbf{R})$, if and only if, for every $x \in U$, one has $N_P(x) \subseteq N_Q(x)$. We say that $K(P)$ is finer than $K(Q)$ if $K(P) \preccurlyeq K(Q)$. We say that $K(P)$ is strictly finer than $K(Q)$, denoted by $K(P) \prec K(Q)$, if $K(P) \preccurlyeq K(Q)$ and $K(P) \neq K(Q)$. In this study, we call $\preccurlyeq$ a rough partial order relation.

3. Geometry in set-based granular computing

3.1. Knowledge distance

In set-based granular computing, information entropy is a main approach to measuring the uncertainty of a granular structure in knowledge bases. If the information entropy of one granular structure is equal to that of the other granular structure, we say that these two granular structures have the same uncertainty. However, it does not mean that these two granular structures are equivalent to each other. In other words, information entropy cannot characterize the difference between any two granular structures in a knowledge base. In this subsection, we introduce a notion of knowledge distance to differentiate any two given granular structures and investigate some of its important properties.

Definition 1. Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$. Knowledge distance between $K(P)$ and $K(Q)$ is defined as

$$D\big(K(P), K(Q)\big) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|}, \tag{2}$$

where $|N_P(x_i) \oplus N_Q(x_i)| = |N_P(x_i) \cup N_Q(x_i)| - |N_P(x_i) \cap N_Q(x_i)|$.

The knowledge distance represents a measure of difference between two granular structures in the same knowledge base. Obviously, $0 \leqslant D(K(P), K(Q)) \leqslant 1 - \frac{1}{|U|}$.

Theorem 1 (*Extremum*). *Let* $K(U, \mathbf{R})$ *be a knowledge base and* $K(P)$ *and* $K(Q)$ *two granular structures on* K. *Then,* $D(K(P), K(Q))$ *achieves its minimum value* $D(K(P), K(Q)) = 0$ *iff* $K(P) = K(Q)$ *and* $D(K(P), K(Q))$ *achieves its maximum value* $D(K(P), K(Q)) = 1 - \frac{1}{|U|}$.

Proof. Given a finite set U, $\forall P, Q \in \mathbf{R}$, one has that $1 \leqslant |N_P(x_i) \cap N_Q(x_i)| \leqslant |U|$ and $1 \leqslant |N_P(x_i) \cup N_Q(x_i)| \leqslant |U|$. Therefore, $\forall P, Q \in \mathbf{R}$, $0 \leqslant |N_P(x_i) \oplus N_Q(x_i)| \leqslant |U| - 1$, i.e., $0 \leqslant \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|} \leqslant 1 - \frac{1}{|U|}$.

If $K(P) = K(Q)$, then $K(P) \cap K(Q) = K(P)$ and $K(P) \cup K(Q) = K(P)$. Hence,

$$D\big(K(P), K(Q)\big) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{0}{|U|} = 0,$$

i.e., $D(K(P), K(Q))$ achieves its minimum value 0 if and only if $K(P) = K(Q)$.

If $K(P) = \wr K(Q)$, then $K(P) \cap K(Q) = K(\omega)$ and $K(P) \cup K(Q) = K(\delta)$. Hence,

$$D\big(K(P), K(Q)\big) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |\{x_i\}|}{|U|} = 1 - \frac{1}{|U|},$$

i.e., $D(K(P), K(Q))$ achieves its maximum value $1 - \frac{1}{|U|}$ if and only if $K(P) = \wr K(Q)$ or $K(Q) = \wr K(P)$.

In particular, one has that $D(K(\omega), K(\delta)) = 1 - \frac{1}{|U|}$. $\quad\square$

From the above extremum, it can easily see that the knowledge distance between a granular structure and its complement is a constant on a given universe.

3.2. Straight-line principle induced by the knowledge distance

Based the discussion in last subsection, we research the geometrical relationships among granular structures with a partial order relation.

From the definition of the partial order relation $\preccurlyeq$ between two granular structures, one knows that the relationship is mainly based on the inclusion relation between neighborhoods in granular structures. Therefore, for the simplicity, we first study this problem from classical sets.

Let A and B be two finite sets, we can measure the difference between two finite sets by the following formula

$$d(A, B) = |A \oplus B|, \tag{3}$$

where $|A \oplus B| = |A \cup B - A \cap B|$.

Lemma 1. *Let A, B, C be three finite sets with $A \subseteq B \subseteq C$ or $A \supseteq B \supseteq C$, then $d(A, B) + d(B, C) = d(A, C)$.*

Proof. Let $A \supseteq B \supseteq C$, thus $A \cup B \cup C = A$ and $B \cup C = B$. Therefore,

$$\begin{aligned}
d(A, B) + d(B, C) &= |A \oplus B| + |B \oplus C| \\
&= \big(|A \cup B - A \cap B|\big) + \big(|B \cup C - B \cap C|\big) \\
&= \big(|A - B|\big) + \big(|B - C|\big) \\
&= |A - C| \\
&= |A \cup C - A \cap C| \\
&= d(A, C).
\end{aligned}$$

For $A \subseteq B \subseteq C$, similarly, one can draw the same conclusion. $\square$

Theorem 2. *Let $K = (U, \mathbf{R})$ be a knowledge base, P, Q, $R \in \mathbf{R}$ and $K(P) \preccurlyeq K(Q) \preccurlyeq K(R)$ or $K(R) \preccurlyeq K(Q) \preccurlyeq K(P)$. Then, $D(K(P), K(R)) = D(K(P), K(Q)) + D(K(Q), K(R))$.*

Proof. For $K(P)$, $K(Q)$, $K(R) \in K$ and $K(P) \preccurlyeq K(Q) \preccurlyeq K(R)$, one can easily get that $N_P(x_i) \subseteq N_Q(x_i) \subseteq N_R(x_i)$, $x_i \in U$. Hence, it follows from Lemma 1 that

$$\begin{aligned}
&D\big(K(P), K(Q)\big) + D\big(K(Q), K(R)\big) \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_Q(x_i) \oplus N_R(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_P(x_i), N_Q(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_Q(x_i), N_R(x_i))}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1}{|U|} \big(d\big(N_P(x_i), N_Q(x_i)\big) + d\big(N_Q(x_i), N_R(x_i)\big)\big) \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_P(x_i), N_R(x_i))}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} D\big(K(P), K(R)\big).
\end{aligned}$$

For $K(R) \preccurlyeq K(Q) \preccurlyeq K(P)$, similarly, one can draw the same conclusion. $\square$

From Theorems 2, one can see that these three granular structures with the partial order relation $\preccurlyeq$ has a sequence relation relative to the knowledge distance. The bigger the value of knowledge distance between the granular structure and the finest knowledge structure, the coarser the granular structure.

As a result of the above discussions and analyses, we come to the following one theorem and three corollaries.

Theorem 3. *Let $\mathbf{K}(U)$ be the set of all granular structures induced by U and $K(P), K(Q) \in \mathbf{K}(U)$ two granular structures. Then, $D(K(P), K(Q)) = D(\wr K(P), \wr K(Q))$.*

Corollary 1. *Let $\mathbf{K}(U)$ be the set of all granular structures induced by U and $K(P), K(Q) \in \mathbf{K}(U)$ two granular structures. If $K(P) \prec K(Q)$, then $D(K(P), K(\omega)) < D(K(Q), K(\omega))$.*

Corollary 2. *Let* $\mathbf{K}(U)$ *be the set of all granular structures induced by U and* $K(P)$, $K(Q) \in \mathbf{K}(U)$ *two granular structures. If* $K(P) \prec K(Q)$, *then* $D(K(P), K(\delta)) < D(K(Q), K(\delta))$.

Corollary 3. *Let* $\mathbf{K}(U)$ *be the set of all granular structures induced by U and* $K(P)$ *a granular structure on* $\mathbf{K}(U)$, *then* $D(K(P), K(\delta)) + D(K(P), K(\omega)) = 1 - \frac{1}{|U|}$.

3.3. Triangle inequality of the knowledge distance

In this subsection, we continue to investigate triangle inequality of the knowledge distance.

From formula (3), we also come to the following lemma.

Lemma 2. *Let A, B, C be three finite sets, then* $d(A, B) + d(B, C) \geqslant d(A, C)$, $d(A, B) + d(A, C) \geqslant d(B, C)$ *and* $d(A, C) + d(B, C) \geqslant d(A, B)$.

Proof. Given A, B, C three finite sets. Let $\forall x \in B \oplus C = (B - C) \cup (C - B)$, then $x \in B - C$ or $x \in C - B$. Hence, it follows that

$$x \in (B - A) \cup (A - C) \quad \text{or} \quad x \in (A - B) \cup (C - A)$$

$$\Rightarrow \quad x \in (B - A) \cup (A - C) \cup (A - B) \cup (C - A)$$

$$\Rightarrow \quad x \in (B - A) \cup (A - B) \cup (A - C) \cup (C - A)$$

$$\Rightarrow \quad x \in (A \oplus B) \cup (A \oplus C)$$

$$\Rightarrow \quad (B \oplus C) \subseteq (A \oplus B) \cup (A \oplus C)$$

$$\Rightarrow \quad |B \oplus C| \leqslant |A \oplus B| + |A \oplus C|.$$

Therefore, one has that

$$d(A, B) + d(A, C) = |A \oplus B| + |A \oplus C| \geqslant |B \oplus C| = d(B, C).$$

Similarly, $d(A, B) + d(A, C) \geqslant d(B, C)$ and $d(A, C) + d(B, C) \geqslant d(A, B)$. $\quad\square$

From this lemma, one can obtain the following theorem.

Theorem 4. *Let* $\mathbf{K}(U)$ *be the set of all granular structures induced by a finite set U, then* $(\mathbf{K}(U), D)$ *is a distance space.*

Proof. (1) One can obtain easily that $D(K(P), K(Q)) \geqslant 0$ from Definition 1.

(2) It is obvious that $D(K(P), K(Q)) = D(K(Q), K(P))$.

(3) For the proof of the triangle inequality principle, one only need to prove that $D(K(P), K(Q)) + D(K(P), K(R)) \geqslant D(K(Q), K(R))$, $D(K(R), K(Q)) + D(K(P), K(R)) \geqslant D(K(Q), K(P))$ and $D(K(R), K(Q)) + D(K(P), K(Q)) \geqslant D(K(P), K(R))$ for any $K(P)$, $K(Q)$, $K(R) \in \mathbf{K}(U)$.

From Lemma 2, we know that for $x_i \in U$, $D(N_P(x_i), N_Q(x_i)) + D(N_P(x_i), N_R(x_i)) \geqslant D(N_Q(x_i), N_R(x_i))$, $D(N_P(x_i), N_Q(x_i)) + D(N_Q(x_i), N_R(x_i)) \geqslant D(N_P(x_i), N_R(x_i))$ and $D(N_P(x_i), N_R(x_i)) + D(N_Q(x_i), N_R(x_i)) \geqslant D(N_P(x_i), N_Q(x_i))$. Hence,

$$
\begin{aligned}
&D\big(K(P), K(Q)\big) + D\big(K(P), K(R)\big) \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_Q(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus N_R(x_i)|}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_P(x_i), N_Q(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_P(x_i), N_R(x_i))}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1}{|U|} \big(d\big(N_P(x_i), N_Q(x_i)\big) + d\big(N_P(x_i), N_R(x_i)\big)\big) \\
&\geqslant \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(N_Q(x_i), N_R(x_i))}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} D\big(K(Q), K(R)\big).
\end{aligned}
$$

Similarly, one can obtain that $D(K(R), K(Q)) + D(K(P), K(R)) \geqslant D(K(Q), K(P))$, $D(K(R), K(Q)) + D(K(P), K(Q)) \geqslant D(K(R), K(P))$.

Therefore, $(\mathbf{K}(U), D)$ is a distance space. □

From the above theorems and discussions, one has further proved the rationality of the proposed knowledge distance. The knowledge distance is a distance measure on all granular structures induced by a finite universe.

4. What is the essence of information granularity

As we know, information granularity, in a broad sense, is the average measure of information granules of a granular structure in a given knowledge base. It can be used to characterize the classification ability of a given granular structure [15,17,31,43,46]. In recent years, some researchers have already started to pay attention to such the problem of what is the essence of information granularity in granular structures. To answer this problem, in this section, we will develop axiomatic definitions of information granularity from three viewpoints in set-based granular computing.

4.1. Two existing axiomatic definitions of information granularity in set-based granular computing

Generally, the partial order relations are often employed for differentiating the coarseness degree between two granular structures in various definitions of information granularity. Liang et al. [15], Bianucci and Gianpiero [5] proposed the partial order relation $\preccurlyeq_1$ to differentiate the coarseness between two partitions in a finite universe.[1] Further, Liang et al. [16,17] gave the partial order relation $\preccurlyeq_2$ to measure the roughness of a family of tolerance classes induced by a tolerance relation. If we adopt the uniform knowledge representation in this study, each of these partial order relations can be induced to a special case of rough partial order relation $\preccurlyeq$.

Using the rough partial order relation $\preccurlyeq$, an axiomatic definition of information granularity was given for differentiating the coarseness between two granular structures in a finite universe [17], which is named a rough granularity.

Definition 2. (See [17].) Let $K = (U, \mathbf{R})$ be a knowledge base, if $\forall P \in \mathbf{R}$, there is a real number $G(P)$ with the following properties:

(1) $G(P) \geqslant 0$ (Non-negative);
(2) $\forall P, Q \in \mathbf{R}$, let $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$, if $K(P) = K(Q)$, then $G(P) = G(Q)$ (Invariability);
(3) if $\forall P, Q \in \mathbf{R}$ and $K(P) \prec K(Q)$, then $G(P) < G(Q)$ (Rough monotonicity).

Then G is called a rough granularity on K.

However, the partial order relation "$\preccurlyeq$" may be not strict in terms of characterizing the properties of information granularity in knowledge bases. It is explained by the following example.

Example 1. Given two granular structures $K(P)$ and $K(Q)$, where

$$K(P) = \big\{ \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_4, x_5, x_6\}, \{x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\} \big\},$$

$$K(Q) = \big\{ \{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4, x_5\},$$
$$\{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_2, x_3, x_5, x_6\} \big\}.$$

Obviously, one has $K(P) \not\preccurlyeq K(Q)$ and $K(Q) \not\preccurlyeq K(P)$. However, intuitively, the granular structure $K(Q)$ should be much coarser than $K(P)$. Unfortunately, we cannot compare the coarseness between these two granular structures using the rough partial order relation $\preccurlyeq$ in this situation.

In order to discover the essence of information granularity, Qian et al. [33] introduced a new binary relation "$\ll$" on $\mathbf{K}(U)$, which is as follows.

Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$. We define a partial order relation $\ll$ as follows:

$K(P) \ll K(Q)$ if and only if, there is a bijective mapping function $f : K(P) \to K(Q)$ such that $|N_P(x)| \leqslant |f(N_Q(x))|$, $x \in U$.

---

[1] In the literature [5], Bianucci and Gianpiero gave four different forms of the partial order relation between partitions. In fact, these four forms are mutually equivalent among them, and the same binary relation can be turned out to be a partial order relation.

Here, we say that $K(P)$ is granulation finer than $K(Q)$ if $K(P) \lll K(Q)$. If there is a bijective mapping function $f : K(P) \rightarrow K(Q)$ such that $|N_P(x)| = |f(N_Q(x))|$, $x \in U$, then it is denoted by $K(P) \approx K(Q)$. If $K(P) \lll K(Q)$ and $K(P) \napprox K(Q)$, we say that $K(Q)$ is strictly granulation coarser than $K(P)$ (or $K(P)$ is strictly granulation finer than $K(Q)$), denoted by $K(P) \ll K(Q)$. In this paper, this partial order relation is called granulation partial order relation.

Using the partial order relation, Qian et al. [33] developed another axiom definition to information granularity, called an information granularity, which is defined as follows.

Definition 3. (See [33].) Let $K = (U, \mathbf{R})$ be a knowledge base, if $\forall P \in \mathbf{R}$, there is a real number $G(P)$ with the following properties:

(1) $G(P) \geqslant 0$ (Non-negativity);
(2) $\forall P, Q \in \mathbf{R}$, let $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$, if there is a bijective mapping function $f : K(P) \rightarrow K(Q)$ such that $|N_P(x)| = |f(N_P(x))|$, then $G(P) = G(Q)$ (Invariability);
(3) if $\forall P, Q \in \mathbf{R}$ and $K(P) \ll K(Q)$, then $G(P) < G(Q)$ (Granulation monotonicity).

Then G is called an information granularity on K.

Example 2. Continued from Example 1. Assume that

$$K'(Q) = \big\{\{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4, x_5\},$$
$$\{x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}\big\}.$$

Obviously, there exists a sequence $K'(Q)$ of $K(Q)$ such that $|N_P(x_i)| \leqslant |N_Q(x_i')|$ ($N_P(x_i) \in K(P)$, $N_Q(x_i') \in K'(Q)$). Since $|N_P(x_3)| = 4 < 6 = |N_Q(x_3')|$, we have that $P \ll Q$. It is clear that the granulation partial order relation has much better performance than the rough partial order relation.

As a result of the above discussions, we come to the following theorems.

Theorem 5. *The rough partial order relation $\preccurlyeq$ is a special instance of the granulation partial order relation $\lll$.*

Proof. Let $K = (U, \mathbf{R})$ be a knowledge base, $P, Q \in \mathbf{R}$, $K(P) = \{N_P(x), \ x \in U\}$ and $K(Q) = \{N_Q(x), \ x \in U\}$. If $K(P) \preccurlyeq K(Q)$, one can obtain that $N_P(x) \subseteq N_Q(x)$ for any $x \in U$, i.e., $|N_P(x)| \leqslant |N_Q(x)|$. That is to say, one can find an array of all neighborhoods in $K(Q)$ such that $K(P) \lll K(Q)$. Therefore, the partial order relation $\preccurlyeq$ is a special instance of the partial order relation $\lll$. $\square$

Theorem 6. *Let $K = (U, \mathbf{R})$ be a knowledge base and $K(P), K(Q)$ two granular structures on K, then $G(P) \leqslant G(Q)$ if $K(P) \preccurlyeq K(Q)$.*

Proof. From the definition of $\preccurlyeq$, one can see that $K(P) \preccurlyeq K(Q)$ ($P, Q \in \mathbf{R}$) if and only if $N_P(x) \subseteq N_Q(x)$, $x \in U$. Hence, for every $N_P(x) \in K(P)$, there exists $N_Q(x) \in K(Q)$ such that $|N_P(x)| \leqslant |N_Q(x)|$, i.e., $K(P) \lll K(Q)$. Therefore, one can easily obtain that $G(P) \leqslant G(Q)$ from Definition 3. $\square$

4.2. Essence of the information granularity is an axiomatic definition induced by the knowledge distance

As we know, information granularity, in a broad sense, is the average measure of information granules of a granular structure in a given knowledge base. It can be used to characterize the discernibility ability of a given granular structure. One of its main tasks is to differentiate the roughness/finess degree between any two granular structures. The rough granularity and the information granularity mentioned in last subsection are used to answer this problem, in which the later is a much better choice. However, this axiomatic definition still has some shortcomings. This is illustrated by the following example.

Example 3. Given two granular structures $K(P)$ and $K(Q)$, where

$$K(P) = \big\{\{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_4, x_5, x_6\}, \{x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}\big\},$$

$$K(Q) = \big\{\{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}\big\}.$$

Obviously, we cannot find any array of members from $K(Q)$ such that $K(P) \lll K(Q)$ and $K(Q) \lll K(P)$. However, intuitively, the granular structure $K(Q)$ should be much coarser than $K(P)$. Unfortunately, we cannot compare the coarseness between these two granular structures using the granulation partial order relation $\ll$ in this situation. In other words, the axiomatic definition of information granularity proposed in previous subsection still not have its limitation for characterizing the coarseness/finess degree of a granular structure.

In this subsection, we will develop a new axiomatic approach of the information granularity using the knowledge distance.

We first understand the meaning of an information granularity from the viewpoint of knowledge distance.

Theorem 7. $D(K(P), K(\omega))$ *is an information granularity.*

Proof. When U is a finite universe, let $K(\omega) = \{[x_i]_\omega \mid [x_i]_\omega = \{x_i\}, x_i \in U\}$ and $K(P) = \{[x_i]_P, \ x_i \in U\}$.

(1) This distance D is clearly non-negative.

(2) If $K(P) \approx K(Q)$, then there is a bijective mapping function $f : K(P) \to K(Q)$ such that $|N_P(x_i)| = |f(N_P(x_i))|$, $x_i \in U$, and $f(N_P(x_i)) = N_Q(x_{j_i})$. One has that

$$
\begin{aligned}
D\big(K(P), K(\omega)\big) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus \{x_i\}|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i)| - 1}{|U|} \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|f(N_P(x_i))| - 1}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_Q(x_{j_i})| - 1}{|U|} \\
&= \frac{1}{|U|} \sum_{j=1}^{|U|} \frac{|N_Q(x_j)| - 1}{|U|} = D\big(K(Q), K(\omega)\big).
\end{aligned}
$$

(3) We prove that if $K(P) \ll K(Q)$, then $D(K(P), K(\omega)) < D(K(Q), K(\omega))$. Let $P, Q \in \mathbf{R}$ with $K(P) \ll K(Q)$, $K(P) = \{N_P(x_1), N_P(x_2), \ldots, N_P(x_{|U|})\}$ and $K(Q) = \{N_Q(x_1), N_Q(x_2), \ldots, N_Q(x_{|U|})\}$, then there exists a sequence $K'(Q)$ of $K(Q)$, where $K'(Q) = \{N_Q(x'_1), N_Q(x'_2), \ldots, N_Q(x'_{|U|})\}$, such that $|N_P(x_i)| < |N_Q(x'_i)|$, and there exists $x_s \in U$ such that $|N_P(x_s)| < |f(N_P(x_s))| = |N_Q(x'_s)|$. Thus,

$$
\begin{aligned}
D\big(K(P), K(\omega)\big) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus \{x_i\}|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i)| - 1}{|U|} \\
&= \frac{1}{|U|} \left(\sum_{i=1, i \neq s}^{|U|} \frac{|N_P(x_i)| - 1}{|U|} + \frac{|N_P(x_s)| - 1}{|U|} \right) \\
&< \frac{1}{|U|} \left(\sum_{i=1, i \neq s}^{|U|} \frac{|N_Q(x_i)| - 1}{|U|} + \frac{|N_Q(x'_s)| - 1}{|U|} \right) \\
&= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_Q(x_i) \oplus \{x_i\}|}{|U|} = D\big(K(Q), K(\omega)\big),
\end{aligned}
$$

i.e., $D(K(P), K(\omega)) < D(K(Q), K(\omega))$.

Summarizing the above, $D(K(P), K(\omega))$ is an information granularity. $\quad\square$

From the above the above theorem, one knows that the knowledge distance between the granular structure $K(P)$ and the finest granular structure $K(\omega)$ can be induced as an information granularity. The distance $D(K(P), K(\omega))$ has much better properties for characterizing information granularity of a granular structure. In the following paragraph, we further explain its advantages.

Using the meaning of the knowledge distance $D(K(P), K(\omega))$, we come back to resurvey the property of information granularity in Definition 3. In fact, the axiomatic definition in Definition 3 is still not the best characterization of information granularity. In Definition 3, one needs to find a suitable mapping function f such that $K(P) \underset{\sim}{\ll} K(Q)$. However, when there does not exist this relationship between $K(P)$ and $K(Q)$, we will not compare their information granularity. From the viewpoint of the knowledge distance, we can overcome this limitation. That is to say, given two granular structures, if one cannot differentiate fineness/roughness relationship between them, we can observe the knowledge distance between each granular structure and the finest granular structure. The bigger the value of knowledge distance between a granular structure and the finest granular structure, and the bigger the information granularity of this granular structure.

Based on the point of view, we develop a more generalized and comprehensible axiomatic definition of information granularity in set-based granular computing.

Definition 4. Let $K = (U, \mathbf{R})$ be a knowledge base, if $\forall P \in \mathbf{R}$, there is a real number $G(P)$ with the following properties:

(1) $G(P) \geqslant 0$ (Non-negative);

(2) $\forall P, Q \in \mathbf{R}$, if $D(K(P), K(\omega)) = D(K(Q), K(\omega))$, then $G(P) = G(Q)$ (Invariability);

(3) $\forall P, Q \in \mathbf{R}$, if $D(K(P), K(\omega)) < D(K(Q), K(\omega))$, then $G(P) < G(Q)$ (Granulation monotonicity).

Then G is called a generalized information granularity on K.

In the literature [15,16,31], several the detailed forms of information granularity have been proposed, such as knowledge granulation and combination granulation. Each of these forms are all induced to one special case of the generalized information granularity.

Theorem 8. *The following properties hold*:

(1) $G(P) = G(\wr \wr P)$;

(2) $G(P \cap Q) \leqslant G(P), G(P \cap Q) \leqslant G(Q)$;

(3) $G(P) \leqslant G(P \cup Q), G(Q) \leqslant G(P \cup Q)$; *and*

(4) $G(P \cap \wr P) = G(\omega), G(P \cup \wr P) = G(\delta)$.

Proof. They are straightforward. $\square$

Example 4. Continued from Example 3. In order to differentiate the coarseness/finess degree between these two granular structures, we calculate two knowledge distance as follows.

$$D\big(K(P), K(\omega)\big) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus \{x_i\}|}{|U|} = \frac{3+3+3+2+2+5}{36} = \frac{18}{36},$$

and

$$D\big(K(Q), K(\omega)\big) = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|N_P(x_i) \oplus \{x_i\}|}{|U|} = \frac{3+3+4+4+4+3}{36} = \frac{21}{36}.$$

Obviously, one has that $D(K(P), K(\omega)) < D(K(Q), K(\omega))$. Hence, the coarseness/finess between these two granular structures can be distinguished. Therefore, the axiomatic definition of generalized information granularity is much better than that of information granularity in Definition 3.

Remark 1. From above analysis and discussions, we draw a conclusion: the axiomatic definition of generalized information granularity defined by the knowledge distance can differentiate the coarseness/finess degree between any two granular structures induced by the same universe, which completely solves the problem of what is the essence of information granularity in set-based granular computing.

Information entropy[2] (also called information measure) and information granularity are two main approaches to measuring the uncertainty of a granular structure [33,35,41]. An information measure can calculate the information content of a granular structure [17,39]. Let $K = (U, \mathbf{R})$ be a knowledge base, $\forall P \in \mathbf{R}$, the information measure $I(P)$ (I is an information measure function) of $K(P)$ should satisfy [17]

(1) $I(P) \geqslant 0$;

(2) if $K(P) = K(Q)$, then $I(P) = I(Q)$;

(3) if $K(P) \prec K(Q)$, then $I(P) > I(Q)$.

For example, each of Shannon's information entropy [39] and Liang's information entropy [15] is an information measure which is used to measure the information content of a complete information system.

In what follows, we establish the relationship between the knowledge distance and an information measure in a knowledge base.

Theorem 9. $D(K(P), K(\delta))$ *is an information measure.*

Proof. When U is a finite universe, let $K(\delta) = \{[x_i]_\delta \mid [x_i]_\delta = U, x_i \in U\}$ and $K(P) = \{[x_i]_P, \; x_i \in U\}$.

---

[2] The concept of entropy comes from classical energetics, which is used to measure out-of-order degree of a system. The entropy of a system as defined by Shannon (1948), gives a measure of uncertainty about its actual structure, which is called information entropy [39]. It has been a useful mechanism for characterizing the information content in various modes and applications in many diverse fields.

(1) This distance D is clearly non-negative.
(2) If $K(P) = K(Q)$, then $D(K(P), K(\delta)) = D(K(Q), K(\delta))$.
(3) We prove that if $K(P) \prec K(Q)$, then $D(K(P), K(\delta)) > D(K(Q), K(\delta))$.

Since the partition $K(P) = \{[x_i]_P \mid [x_i]_P = U, x_i \in U\}$ and $K(P) \prec K(Q)$, so $[x_i]_P \subseteq [x_i]_Q \subseteq U$, $x_i \in U$, and there exists $x_0 \in U$ such that $[x_0]_P \subset [x_0]_Q$. Hence,

$$
\begin{aligned}
D\big(K(P), K(\delta)\big) &= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|[x_i]_P \oplus U|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |[x_i]_P|}{|U|} \\
&> \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|U| - |[x_i]_Q|}{|U|} = \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|[x_i]_Q \oplus U|}{|U|} \\
&= D\big(K(Q), K(\delta)\big).
\end{aligned}
$$

That is $D(K(P), K(\delta)) > D(K(Q), K(\delta))$.

Summarizing the above, $D(K(P), K(\delta))$ is an information measure. □

Remark 2. From the above analysis, we can briefly understand information measure and information granularity by the knowledge distance. Given a granular structure, the bigger the value of the knowledge distance between this granular structure and the finest granular structure is, the smaller the information content of this granular structure is, and the bigger the information granularity of this granular structure is. In other words, the knowledge distance establishes the relationship between information measure and information granularity, which provides a more comprehensible perspective for uncertainty of a granular structure in set-based granular computing.

4.3. A lattice model to the generalized information granularity

Zadeh pointed out in the literature [54], "In general, information granularity should characterize the granulation degree of objects from the viewpoint of hierarchy". This provides a point of view that an information granularity should characterize hierarchical relationships among granular structures. In the following, we analyze the hierarchical relationships through using a lattice model.

In order to construct a lattice structure, we first introduce three generalized information granularity operators, which is as follows.

Definition 5. Let $G(U)$ be the set including all information granularity induced by U, and $G(P), G(Q) \in G(U)$ two generalized information granularity. Three granularity operators $\sqcap$, $\sqcup$, and $\wr$ on $G(U)$ are defined as

$$G(P) \sqcap G(Q) = \min\big\{G(P), G(Q)\big\}, \tag{4}$$

$$G(P) \sqcup G(Q) = \max\big\{G(P), G(Q)\big\}, \tag{5}$$

$$\wr G(P) = G(\wr P). \tag{6}$$

Theorem 10. Let $\sqcap$, $\sqcup$ be two granularity operators, then

(1) $G(P) \sqcap G(P) = G(P)$, $G(P) \sqcup G(P) = G(P)$;
(2) $G(P) \sqcap G(Q) = G(Q) \sqcap G(P)$, $G(P) \sqcup G(Q) = G(Q) \sqcup G(P)$;
(3) $G(P) \sqcap (G(P) \sqcup G(Q)) = G(P)$, $G(P) \sqcup (G(P) \sqcap G(Q)) = G(P)$; and
(4) $(G(P) \sqcap G(Q)) \sqcap G(R) = G(P) \sqcap (G(Q) \sqcap G(R))$, $(G(P) \sqcup G(Q)) \sqcup G(R) = G(P) \sqcup (G(Q) \sqcup G(R))$.

Proof. They are straightforward from Definition 5. □

Theorem 11. Let $\sqcap$, $\sqcup$, and $\wr$ be three operators on $G(U)$, then

(1) $\wr(\wr G(P)) = G(P)$,
(2) $G(P) \sqcap G(\wr P) = 0$, $G(P) \sqcup G(\wr P) = 1 - \frac{1}{|U|}$,
(3) $\wr(G(P) \sqcap G(Q)) = \wr G(P) \sqcup \wr G(Q)$, and
(4) $\wr(G(P) \sqcup G(Q)) = \wr G(P) \sqcap \wr G(Q)$.

Proof. They can be proved using the formulas (4), (5) and (6).

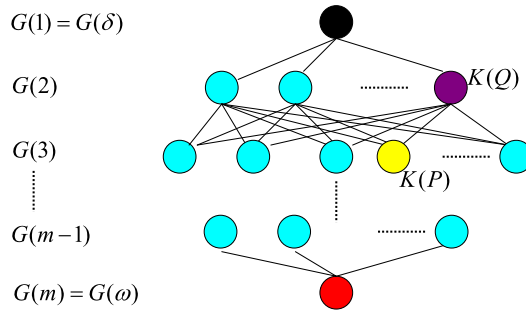Theorem 11 indicates that (1) is reflexive, (2) is complementary, and (3) and (4) are two dual principles. □

Fig. 1. The lattice structure of generalized information granularity.

Theorem 12. *Let $\sqcap$, $\sqcup$ and $\wr$ be three granularity operators, the following properties hold*:

(1) *If $G(P) \leqslant G(Q)$, then $\wr G(Q) \leqslant \wr G(P)$;*
(2) *$G(P) \sqcap G(Q) \leqslant G(P)$, $G(P) \sqcap G(Q) \leqslant G(Q)$; and*
(3) *$G(P) \leqslant G(P) \sqcup G(Q)$, $G(Q) \leqslant G(P) \sqcup G(Q)$.*

Proof. They are straightforward from Definition 4 and Definition 5. □

Theorem 13. *Let $G(U)$ be the set including all information granularity induced by U. Then, $(G(U), \leqslant)$ is a poset.*

Proof. Let $G(P)$, $G(Q)$ and $G(R)$ be three information granularity in $G(U)$.

(1) For arbitrary $G(P)$, one has that $G(P) \leqslant G(P)$.
(2) Suppose that $G(P) \leqslant G(Q)$ and $G(Q) \leqslant G(P)$, it is obvious that $G(P) = G(Q)$.
(3) Suppose that $G(P) \leqslant G(Q)$ and $G(Q) \leqslant G(R)$, we have that $G(P) \leqslant G(R)$.

From the above three items, it follows that $(G(U), \leqslant)$ is a poset. □

Theorem 14. *$(G(U), \sqcap, \sqcup, \wr)$ is a distributive lattice.*

Proof. At first, we prove $(G(U), \sqcap, \sqcup)$ is a lattice.
From (2) and (4) in Theorem 11, the commutative law and associative law in the lattice definition are obvious.
Let $G(P), G(Q), G(R) \in G(U)$ be three generalized information granularity, one can obtain that

$$G(P) \sqcap G(Q) = G(P)$$
$$\iff \quad \min\{G(P), G(Q) = G(P)$$
$$\iff \quad G(P) \leqslant G(Q).$$

According to the duality principle in a lattice, one can easily get that $G(P) \sqcup G(Q) = G(P) \iff G(Q) \leqslant G(P)$.
Furthermore, for $G(P), G(Q), G(R) \in G(U)$, we know that

$$G(P) \sqcap \big(G(Q) \sqcup G(R)\big) = \big(G(P) \sqcap G(Q)\big) \sqcup \big(G(P) \sqcap G(R)\big), \quad \text{and}$$
$$G(P) \sqcup \big(G(Q) \sqcap G(R)\big) = \big(G(P) \sqcup G(Q)\big) \sqcap \big(G(P) \sqcup G(R)\big).$$

Therefore, $(K, \sqcup, \sqcap)$ is a distributive lattice.
Hence, $(G(U), \sqcap, \sqcup, \wr)$ is a distributive lattice. □

To more easily understand the lattice model induced by the generalized information granularity, we employ Fig. 1 for further illustration.

In Fig. 1, $G(i)$ $(i \leqslant m)$ means the i-th layer in this lattice, where $G(1) = G(\delta)$ is the biggest information granularity, and $G(m) = G(\omega)$ is the smallest information granularity. From bottom to top (from small to big) according to the value of generalized information granularity (i.e., knowledge distance $D(K(P), K(\omega))$), those generalized information granularity with the same value are put on the same layer of the lattice. In fact, the lattice induced by generalized information granularity is a chain (a kind of special lattice).

In the following, we employ an example with $U = \{x_1, x_2, x_3\}$ for further illustrating the lattice in Fig. 1.
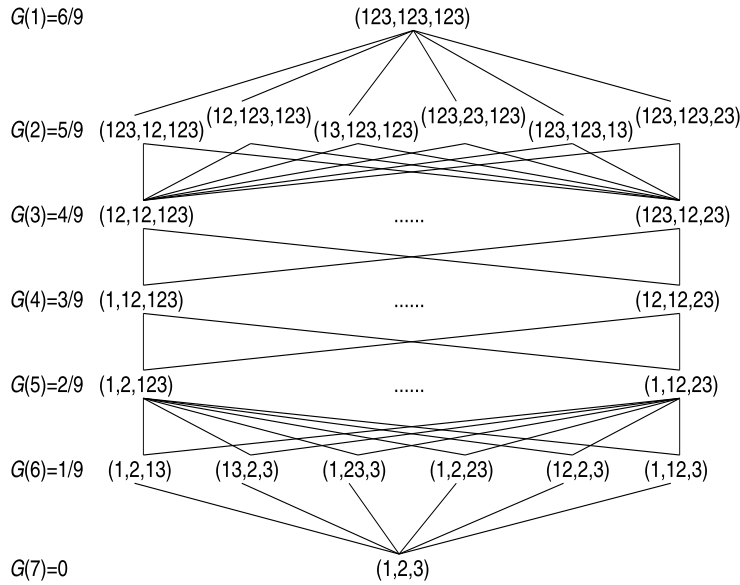
Fig. 2. The lattice structure of generalized information granularity.

Example 5. Let $U = \{x_1, x_2, x_3\}$. We give the lattice structure induced by all generalized information granularity from U, see Fig. 2.

5. How to perform granular operation

In set-based granular computing, granular operation is to answer the problem how to achieve composition, decomposition and transformation of information granules/granular structures. This problem is also one of key tasks in set-based granular computing [33,44,47,51,53]. In this section, we will propose four operators for achieving composition, decomposition and transformation of granular structures, and investigate the algebra structure of granular structures under these operators.

5.1. Operators for granular structures: composition, decomposition and transformation

There are two types of operators to be considered in set-based granular computing. One is operations among information granules, the other is operations among granular structures in a knowledge base. As operations among information granules is based on classical sets, we still operate on them by $\cap$, $\cup$, $-$ and $\sim$, i.e., a new information granule can be generated by $\cap$, $\cup$, $-$ and $\sim$ on known information granules. However, operations among granular structures are performed through composing and decomposing known granular structures in knowledge bases in essence. Therefore, the operators on a knowledge base to generate new granular structures are very desirable. In the following, we introduce four operators among granular structures in a knowledge base.

Definition 6. Let $K = (U, \mathbf{R})$ be a knowledge base and $K(P)$, $K(Q) \in K$ two granular structures. Four operators $\cap$, $\cup$, $-$ and $\wr$ on K are defined as

$$K(P) \cap K(Q) = \left\{ N_{P \cap Q}(x) = N_P(x) \cap N_Q(x), x \in U \right\}, \tag{7}$$

$$K(P) \cup K(Q) = \left\{ N_{P \cup Q}(x) = N_P(x) \cup N_Q(x), x \in U \right\}, \tag{8}$$

$$K(P) - K(Q) = \left\{ N_{P-Q}(x) = x \cup \left(N_P(x) - N_Q(x) \right), x \in U \right\}, \tag{9}$$

$$\wr K(P) = \left\{ \wr N_P(x) = x \cup {\sim} N_P(x), x \in U \right\}, \tag{10}$$

where ${\sim} N_P(x) = U - N_P(x)$.

Note. The proposed four operators can be seen as intersection operation, union operation, subtraction operation and complement operation in-between granular structures, which are used to fine, coarsen, decompose granular structures and calculate complement of a granular structure, respectively. In Definition 6, $\cap$ and $\cup$ operators can be used to compose two granular structures to a new granular structure, where one can get a much finer granular structure using $\cap$ operator, and one can form a much coarser one using $\cup$ operator. The operator $-$ can be understood as a decomposition mechanism and

be used to generate much finer granular structures. The operator $\wr$ may be viewed as one of mappings between granular structures, which can transform one granular structure into another granular structure.

Here, we regard $\cap$, $\cup$, $-$ and $\wr$ as four atomic formulas and finite connection on them are all formulas. Through using these operators, one can obtain a new granular structure via some known granular structures on U. Let $\mathbf{K}(U)$ denote the set of all granular structures on U, then these four operators $\cap$, $\cup$, $-$ and $\wr$ on $\mathbf{K}(U)$ are close. As follows, we investigate several fundamental algebra properties of these four operators.

Theorem 15. *Let* $\cap$, $\cup$ *be two operators on K, then*

(1) $K(P) \cap K(P) = K(P)$, $K(P) \cup K(P) = K(P)$;
(2) $K(P) \cap K(Q) = K(Q) \cap K(P)$, $K(P) \cup K(Q) = K(Q) \cup K(P)$;
(3) $K(P) \cap (K(P) \cup K(Q)) = K(P)$, $K(P) \cup (K(P) \cap K(Q)) = K(P)$; *and*
(4) $(K(P) \cap K(Q)) \cap K(R) = K(P) \cap (K(Q) \cap K(R))$, $(K(P) \cup K(Q)) \cup K(R) = K(P) \cup (K(Q) \cup K(R))$.

Proof. They are straightforward from Definition 6.

Theorem 16. *Let* $\cap$, $\cup$ *and* $\wr$ *be three operators on K, then*

(1) $\wr(\wr K(P)) = K(P)$,
(2) $K(P) \cap \wr K(P) = \{x \mid x \in U\}$,
(3) $\wr(K(P) \cap K(Q)) = \wr K(P) \cup \wr K(Q)$, *and*
(4) $\wr(K(P) \cup K(Q)) = \wr K(P) \cap \wr K(Q)$.

Proof. For any $x \in U$, $K(P)$, $K(Q) \in K$, $N_P(x)$ is the neighborhood class induced by x in $K(P)$.

(1) From Definition 6, one can easily see that $\wr(N_P(x)) = x_i \cup \sim N_P(x)$ and $\wr(\wr(N_P(x))) = x \cup (x \cup N_P(x)) = N_P(x)$. Therefore, $\wr(\wr K(P)) = K(P)$.

(2) From Definition 6, it follows that $N_P(x) \cap \wr(N_P(x)) = x$, $\forall x \in U$. Then, $K(P) \cap \sim K(P) = \{x, \ x \in U\}$.

(3) According to Definition 6, $\forall x \in U$, it follows that

$$\wr\big(N_P(x) \cap N_Q(x)\big) = x_i \cup \sim\big(N_P(x) \cap N_Q(x)\big)$$
$$= x \cup \big(\sim N_P(x) \cup \sim N_Q(x)\big)$$
$$= \big(x \cup \sim N_P(x)\big) \cup \big(x \cup \sim N_Q(x)\big)$$
$$= \wr N_P(x) \cup \wr N_Q(x).$$

Therefore, one can get that $\wr(K(P) \cap K(Q)) = \wr K(P) \cup \wr K(Q)$.

(4) According to Definition 6, $\forall x \in U$, one has that

$$\wr\big(N_P(x) \cup N_Q(x)\big) = x_i \cup \sim\big(N_P(x) \cup N_Q(x)\big)$$
$$= x \cup \big(\sim N_P(x) \cap \sim N_Q(x)\big)$$
$$= \big(x \cup \sim N_P(x)\big) \cap \big(x_i \cup \sim N_Q(x)\big)$$
$$= \wr N_P(x) \cap \wr N_Q(x).$$

Hence, one can obtain that $\wr(K(P) \cup K(Q)) = \wr K(P) \cap \wr K(Q)$. $\square$

Theorem 16 shows that (1) is reflexive, (2) is complementary, and (3) and (4) are two dual principles.

Theorem 17. *Let* $\cap$, $\cup$, $-$ *and* $\wr$ *be operators on K, then*

(1) $K(P) - K(Q) = K(P) \cap \wr K(Q)$,
(2) $K(P) - K(Q) = K(P) - (K(P) \cap K(Q))$,
(3) $K(P) \cap (K(Q) - K(R)) = (K(P) \cap K(Q)) - (K(P) \cap K(R))$, *and*
(4) $(K(P) - K(Q)) \cup K(Q) = K(P)$.

Proof. They are straightforward from Definition 6. $\square$

Suppose $K = (U, R)$ be a knowledge base, P, $Q \in R$, and $K(P)$, $K(Q) \in K$ be two granular structures induced by P, Q, respectively. To investigate properties of the operations among granular structures on a knowledge base, we develop the following theorem.

Theorem 18. *Let $\cap$, $\cup$ and $\wr$ be three operators on K, the following properties hold*:

(1) *If $K(P) \preccurlyeq K(Q)$, then $\wr K(Q) \preccurlyeq \wr K(P)$*;
(2) $K(P) \cap K(Q) \preccurlyeq K(P)$, $K(P) \cap K(Q) \preccurlyeq K(Q)$; *and*
(3) $K(P) \preccurlyeq K(P) \cup K(Q)$, $K(Q) \preccurlyeq K(P) \cup K(Q)$.

Proof. The terms (2) and (3) can be easily proved from (11) and (12) in Definition 6, respectively.
From Definition 6, one can obtain that

$$
\begin{aligned}
K(P) \preccurlyeq K(Q) \quad &\Longrightarrow \quad \forall x \in U, \quad N_P(x) \subseteq N_Q(x) \\
&\Longrightarrow \quad \forall x \in U, \quad \sim N_Q(x) \subseteq \sim N_P(x) \\
&\Longrightarrow \quad \forall x \in U, \quad x \cup \sim N_Q(x) \subseteq x \cup \sim N_P(x) \\
&\Longrightarrow \quad \wr K(Q) \preccurlyeq \wr K(P).
\end{aligned}
$$

Hence, the term (1) in this theorem holds. $\square$

Theorem 19. *Let $K = (U, \mathbf{R})$ be a knowledge base, $\forall P, Q \in \mathbf{R}$. Then,*

$$
G(P) + G(Q) = G(P \cup Q) + G(P \cap Q).
$$

Proof. It is straightforward. $\square$

5.2. A lattice model on granular structures

As mentioned above, in the viewpoint of knowledge engineering, the operators on granular structures to generate new granular structures are very significant. These granular structures generated provide indispensable knowledge and find a basis in human reasoning based on a family of granular structures, in which there is an underlying algebra structure. In this subsection, we reveal the underlying algebra structure.

Theorem 20. $(K, \cup, \cap)$ *is a distributive lattice.*

Proof. At first, we prove $(K, \cup, \cap)$ is a lattice.
From (2) and (4) in Theorem 15, the commutative law and associative law in the lattice definition are obvious.
Let $K(P), K(Q), K(R) \in K$ be three granular structures, where $K(P) = \{N_P(x), \ x \in U\}$, $K(Q) = \{N_Q(x), \ x \in U\}$ and $K(R) = \{N_R(x), \ x \in U\}$. one can obtain that

$$
\begin{aligned}
K(P) \cap K(Q) &= K(P) \\
\Longleftrightarrow \quad &\forall x \in U, \quad N_{P \cap Q}(x) = N_P(x), \quad x \in U \\
\Longleftrightarrow \quad &N_P(x) \cap N_Q(x) = N_P(x) \\
\Longleftrightarrow \quad &N_P(x) \subseteq N_Q(x), \quad \forall x \in U \\
\Longleftrightarrow \quad &K(P) \preccurlyeq K(Q).
\end{aligned}
$$

According to the duality principle in a lattice, one can easily get that $K(P) \cup K(Q) = K(P) \Longleftrightarrow K(Q) \preccurlyeq K(P)$.
In addition, for $K(P), K(Q), K(R) \in K$, we know that

$$
N_P(x) \cap \big(N_Q(x) \cup N_R(x)\big) = \big(N_P(x) \cap \big(N_Q(x)\big)\big) \cup \big(N_P(x) \cap N_R(x)\big), \quad \forall x \in U.
$$

Hence, $K(P) \cap (K(Q) \cup K(R)) = (K(P) \cap K(Q)) \cup (K(P) \cap K(R))$.
From the duality principle in a lattice, one can get that $K(P) \cup (K(Q) \cap K(R)) = (K(P) \cup K(Q)) \cap (K(P) \cup K(R))$. Therefore, $(K, \cup, \cap)$ is a distributive lattice. $\square$

Theorem 21. *Let $\mathbf{K}(U)$ be the set of all granular structures on U, then $(\mathbf{K}(U), \cup, \cap, \wr)$ is a complemented lattice.*

Proof. From Theorem 20, it is obvious that $(\mathbf{K}(U), \cup, \cap, \wr)$ is a distributive lattice. Furthermore, from (1) in Theorem 15, one can get that $\wr(\wr K(P)) = K(P)$. In addition, from (10) in Definition 6, one has that
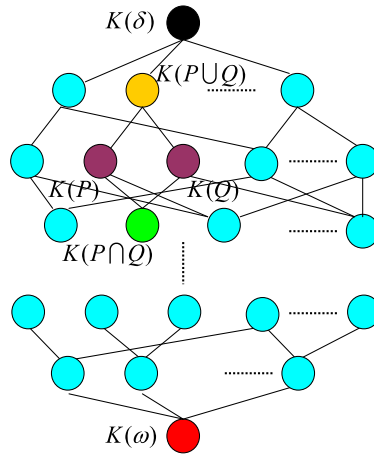
Fig. 3. The lattice structure of granular structures.

$$K(P) \preccurlyeq K(Q) \iff \forall x \in U, \quad N_P(x) \subseteq N_Q(x)$$
$$\iff \forall x \in U, \quad \sim N_P(x) \supseteq \sim N_Q(x)$$
$$\iff \forall x \in U, \quad x \cup \sim N_P(x) \supseteq x \cup \sim N_Q(x)$$
$$\iff \forall x \in U, \quad \wr N_P(x) \supseteq \wr N_Q(x)$$
$$\iff \wr K(Q) \preccurlyeq \wr K(P).$$

In addition, we know that $\forall K(P)$, there must exist its complemented structure $\wr K(P)$ such that $K(P) \cup \wr K(P) = K(\delta)$ and $K(P) \cap \wr K(P) = K(\omega)$, where $K(\delta)$ and $K(\omega)$ can be seen as the entire upper bound and the entire lower bound of this lattice, respectively.

Hence, $(\mathbf{K}(U), \cup, \cap, \wr)$ is a complemented lattice. □

Fig. 3 gives the sketch map of the lattice on all granular structures from a finite universe.

In Fig. 3, for arbitrary two granular structures $K(P)$ and $K(Q)$ in the complemented lattice, their supremum (least upper bound) is uniquely determined by $K(P) \cup K(Q)$, and their infimum (greatest lower bound) is also uniquely determined by $K(P) \cap K(Q)$. In the complemented lattice $(\mathbf{K}(U), \cup, \cap, \wr)$, the granular structure $K(\omega) = \{x \mid x \in U\}$ and the granular structure $K(\delta) = \{N_P(x) \mid N_P(x) = U, x_i \in U\}$ are two special granular structures, where $K(\omega)$ is the discrete classification and $K(\delta)$ is the indiscrete classification. For any $K(P) \in \mathbf{K}(U)$, one has that $K(\omega) \preccurlyeq K(P) \preccurlyeq K(\delta)$. Then, we can call $K(\omega)$ and $K(\delta)$ the minimal element and the maximal element on the lattice $(\mathbf{K}(U), \cup, \cap, \wr)$, respectively.

In the following, we continue by Example 5 for further illustrating the lattice in Fig. 3.

Example 6. Let $U = \{x_1, x_2, x_3\}$. We give the lattice structure induced by all granular structures from U, see Fig. 4.

From the above analyses and discussions, it follows that the lattice model and four operators on it provides a framework for composition, decomposition and transformation among granular structures in set-based granular computing, which solve the problem of how to perform granular operation in set-based granular computing.

In granular computing, there are several studies on levels of granularity [1–3,52]. In particular, Yao [52] studied the multi-level abstraction view of granular computing in discussion of integrative levels of granularity, which is opposed to multi-level aggregation of data. It is deserved to point out that these results and the lattice model induced by the proposed four operators in this study is compatible. In the lattice induced by the proposed four operators, all granular structures can be found, which can characterize the relationships among these granular structures induced by a given universe.

The proposed four operators on granular structures have some potential applications in data mining and knowledge discovery. For example, as we know, one of the strengths of rough set theory is the fact that an unknown target concept can be characterized approximately by existing granular structures in a knowledge base. From the above analyses, it is shown that these four operators ($\cup$, $\cap$, $\wr$ and $-$) can be applied to generate new granular structures on a knowledge base.

Example 7. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Given $K(P) = \{\{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_4, x_5, x_6\}, \{x_4, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}\}$ and $K(Q) = \{\{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_5, x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4, x_5\}, \{x_1, x_2, x_3, x_4, x_5, x_6\}, \{x_2, x_3, x_5, x_6\}\}$ two granular structures, and a target concept $X = \{x_1, x_3, x_5, x_6\}$.
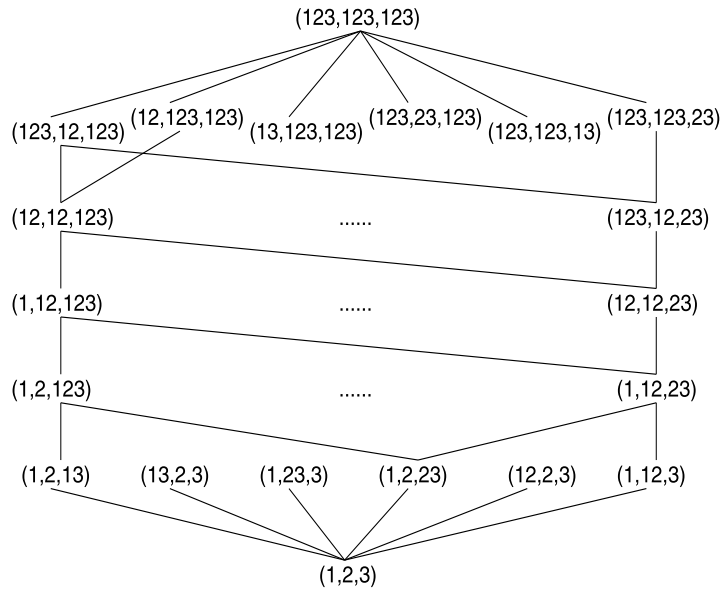
Fig. 4. The lattice structure of generalized information granularity.

We use the interaction operator $\cap$ to generate a new granular structure. Then,

$$K(P) \cap K(Q) = \Big\{ \{x_1, x_3\}, \{x_2, x_3, x_6\}, \{x_1, x_2, x_3, x_6\}, \{x_2, x_3, x_5, x_6\}, \{x_5, x_6\}, \{x_4, x_5, x_6\} \Big\}.$$

Through computing lower approximation of X, we have that

$$\underline{P}(X) = \varnothing,$$
$$\underline{Q}(X) = \varnothing, \quad \text{and}$$
$$\underline{P \cap Q}(X) = \{x_1, x_3, x_5, x_6\}.$$

Hence, their approximation accuracies are

$$apr_P(X) = 0,$$
$$apr_Q(X) = 0, \quad \text{and}$$
$$apr_{P \cap Q}(X) = 1.$$

Therefore, one has that $apr_{P \cap Q}(X) > apr_P(X)$ and $apr_{P \cap Q}(X) > apr_Q(X)$.

From the above example, it can be seen that one can use these new granular structures to approximate an unknown target with much better approximation accuracy. Therefore, this mechanism may be used to rule extraction and knowledge discovery from knowledge bases.

6. Conclusions

To obtain a unified framework for set-based granular computing, through consistently representing granular structures, we have first introduced a concept of knowledge distance and then formulated a lattice model based on it. In the lattice model, a so-called axiomatic definition to generalized information granularity has been developed, which have enabled us to solve one of the fundamental problems: what is the essence of information granularity in set-based granular computing? Moreover, to tackle another fundamental problem of how to perform granular operation, four operators have been presented on granular structures, under which the algebraic structure of granular structures is a complementary distributive lattice. These operators can effectively achieve composition, decomposition and transformation of granular structures. These results show that the knowledge distance and the lattice model are powerful mechanisms for studying set-based granular computing.

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