

# An adaptive consensus method for multi-attribute group decision making under uncertain linguistic environment



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## ABSTRACT

For a multi-attribute group decision making (MAGDM) problem, the so-called consensus reaching process is used to achieve an agreement among experts and finally make a common decision. Unfortunately, so far the consensus models for MAGDM haven't been completely studied, especially for MAGDM under uncertain linguistic environment. The disadvantages of most existing consensus models could be summarized into 3 aspects. (1) In most existing consensus models, all the experts' opinions are weighted equally important, and/or all the experts' weights are treated statically. (2) Most of the interactive consensus methods are lack of effective feedback mechanism, while the automatic ones also have some defects, such as the lack of pertinence in adjustment process and the inability to reflect the subjective opinions of experts. (3) Also the comparison methods for uncertain linguistic variables therein are far from perfect, which require either complicated computing process or may cause non-distinguishable cases. In order to solve the above problems and obtain final decision results more efficiently, an interactive method with adaptive experts' weights and explicit guidance rules for MAGDM under uncertain linguistic environment is developed. Our contributions can be summarized as follows. (1) Based on the definitions of closeness and consensus indices, a non-linear programming model is constructed to dynamically adjust the experts' weights by maximizing the group consensus. (2) A targeted feedback mechanism including identification rules and recommendation rules is designed to guide the experts to modify their opinions more precisely and effectively. (3) A more appropriate method for comparing uncertain linguistic variables named dominance index is proposed, which can simplify the calculation process significantly. Finally, an illustrative example proves that the proposed consensus method is feasible and effective, and a detailed comparison and analysis highlights the advantages and characteristics of this method.

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## 1. Introduction

Group decision making (GDM) is a type of participatory process where the best solution is selected among the alternatives by taking into account individual opinions of multiple experts [1,2]. As an important research direction of modern science, GDM has been extensively studied and already proven to be valuable in various areas such as society, economy, management and engineering [3–7]. The essential target in GDM is to find a solution accepted by all the experts in a group, i.e., to reach a broad consensus among experts [8,9]. However, a consensus is usually impacted

by many nonideal factors in practice. For example, experts coming from different areas have individual knowledge structures, expression abilities, evaluation levels and preferences as well as practical experience, so that they may have different opinions for the same problem and different perceptions for the importance of various factors therein. Consequently, if the decision maker integrates various experts' opinions straightforwardly, it is hard to get a final result as expected. To solve this issue, dynamic and iterative consensus-reaching methods were proposed and are becoming more and more important during the whole GDM process and so have attracted wide attentions from researchers.

To the best of our knowledge, the existing consensus-reaching methods for GDM problems could be classified into several categories: (1) methods based on individual consistency measures [10–17]; (2) methods based on soft consensus measures [18,19]. Different soft consensus measures have been developed

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under different environments, such as fuzzy preference relations [3,20], multi-granular linguistic information [21,22], linguistic or uncertain linguistic preference relations [23], uncertain linguistic preference relations based on 2-tuples [24,25] and linguistic interval fuzzy preference relations [26]; (3) methods to satisfy different application requirements. Such as mobile decision support system [1], GDM model for evaluating supply chain flexibility [27], GDM problems with non-homogeneous experts [28], autocratic GDM problems [29], Web 2.0 communities involving large numbers of users [30], heterogeneous GDM problems [31], large-scale GDM with noncooperative behaviors [32], large-scale GDM problems with heterogeneous information [33] and large group emergency decision making [34].

However, the above methods are mainly used to solve general GDM problems which are based on preference relations. Different from general GDM, MAGDM compares a set of alternatives by multiple experts under multiple attributes based on the decision matrices provided by experts so that the decision maker can rank all the alternatives and select the best one(s) [35–40]. But similar to general GDM, during solve the MAGDM problems, the consensus reaching process is still required to reach an agreement among experts before making a common decision.

Currently, popular consensus methods for MAGDM problems can be divided into the interactive ones and the automatic ones. The former needs to return the individual opinions with larger deviation from the group opinions to the corresponding experts and invite them to reevaluate the alternatives until a high consensus is reached [41–47]. Xu [41] developed an interactive approach to MAGDM with multiple sources of uncertain linguistic information assessed in different unbalanced linguistic label sets. Fu and Yang [42] extended the evidential reasoning approach to group consensus situations for MAGDM problems based on utilities. They [43] further proposed an evidential reasoning based consensus model for MAGDM problems with interval-valued group consensus requirements. Parreiras et al. [44] proposed a flexible consensus scheme for GDM, which can obtain a consistent collective opinion from information provided by each expert in terms of multigranular fuzzy estimates. Singh and Benyoucef [45] presented a fuzzy TOPSIS and soft consensus based GDM methodology to solve the multi-criteria decision making problems in supply chain coordination. Su et al. [46] developed an interactive method to solve the dynamic intuitionistic fuzzy MAGDM problems, in which all the attributes values provided by multiple decision makers at different periods take the form of intuitionistic fuzzy numbers. Zhang and Li et al. [47] developed a consensus model for MAGDM under intuitionistic fuzzy environment. Xu and Du et al. [48] established a consensus model for multi-criteria large-group emergency decision making considering non-cooperative behaviors and minority opinions. The interactive methods rely on the experts' advice closely which makes them relatively reliable and accurate, but most of the methods are lack of effective feedback mechanisms to guide the experts to make quick adjustments. That is, in the adjustment process the experts may have a certain blindness. In most cases, they can neither get any adjustment suggestions nor grasp the direction or degree of adjustment well. So in many practical MAGDM problems, the interactive methods are time-consuming and expensive.

The automatic methods can automatically modify the individual opinions and make the group opinions reach consensus, which not only avoids the troublesome procedures, but also saves time and improves efficiency. Most of the existing automatic methods are developed based on crisp numbers [49–51]. Xu [49] aggregated the individual decision matrices into a group decision matrix by using the additive weighted aggregation (AWA) operator, and then established a convergent iterative algorithm. Zhang [50] proposed two optimization-based consensus rules for MAGDM problems, called consensus rules with minimum adjustments. Xu and Wu [51]

developed a discrete support model to automatically modify the preferences of some experts. In order to solve the MAGDM problems under uncertain linguistic environment, Xu and Wu et al. [52] further proposed a new consensus based support model by defining the superiority indices. But the automatic methods are lack of pertinence and cannot reflect the subjective adjustment opinions from the experts during the consensus reaching process.

In general, the above research work has expanded the applicable scope of the consensus model and significantly advanced the field of MAGDM. To better illustrate the above consensus models in detail, a comparative analysis is made and summarized in Table 1.

Through the above analysis of the relevant literatures, we can see that compared with GDM the study of consensus models for MAGDM is still relatively weak and needs to be further strengthened and deepened. If one can build a new consensus model by exploiting the advantages of both interactive methods and automatic methods, better final decision results are able to be found more efficiently. In addition, in many alternative selection problems, due to the estimation inaccuracies, lack of knowledge, and limited expertise in the problem domain, experts are more inclined to use uncertain linguistic information, such as "between good and very good", to express their preference. That is why the MAGDM problems under uncertain linguistic environment are widespread and play an important role in the real life [52,53], but the consensus methods specifically for them are relatively insufficient. Therefore, we focus on the MAGDM with uncertain linguistic information and discuss its consensus problem in this paper.

Comparing with the existing consensus models for MAGDM problems, our contributions can be mainly summarized in the following four aspects:

- (1) In most of the existing consensus methods, the consensus measures belong to category 2 (see notes below Table 1). Such methods have to compare the individual decision matrix provided by each expert with the group decision matrix aggregated by all experts' individual decision matrices [41,44–52], which may change the group decision matrix due to the adjustment of individual decision matrix. In order to reduce the impact of individual opinions on the group and avoid the aggregation process, certain consensus indices based on the closeness between experts are provided, which belong to category 1 (see notes below Table 1).
- (2) Most of the existing consensus models weight all the experts' opinions equally important or treat all the experts' weights statically in the whole consensus reaching process. That is, the experts' weights do not change with the adjustment of their opinions. To maximize group consensus and reach a more rational final solution, the impact of the modified opinions on the experts' weights should be considered and the experts' weights should adapt accordingly.
- (3) Most of the interactive consensus methods for MAGDM are lack of effective feedback mechanism, and the adjustments made by the automatic ones are lack of pertinence and cannot reflect the subjective opinions of the experts. In order to reduce the blindness of adjustment and accelerate the speed of group consensus, a targeted feedback mechanism with explicit guiding rules is proposed to help the experts modify their opinions more effectively.
- (4) The existing comparison methods for uncertain linguistic variables, such as the possibility degree [41,54–59] and the superiority index [52], have some limitations. For example, the computation process is complicated and in some cases there are two uncertain linguistic variables that cannot be distinguished, so a new method is proposed to efficiently compare the uncertain linguistic variables.

**Table 1**  
Comparison of different consensus methods for MAGDM.

Methods	Evaluation information	Consensus measure	Feedback mechanism	Guiding rule	Experts' weights
Xu [49]	Numbers	Category 2	Automatic, global	Not given	Static
Zhang and Dong [50]	Numbers	Category 2	Automatic, global	Consensus rules	Not mentioned
Xu and Wu [51]	Numbers	Category 2	Automatic, local	Find the farthest elements from the mean	Static
Xu and Wu et al. [52]	ULV	Category 2	Automatic, global	Not given	Static
Xu [41]	ULI/MULI	Category 2	Interactive, local	Not given	Static
Su et al. [46]	IFNs	Category 2	Interactive, local	Not given	Static
Parreiras et al. [44]	MLT	Category 2	Interactive, local	Not given	Adjusted by the moderator
Fu and Yang [42]	Utilities	Category 1	Interactive, local	Not given	Not mentioned
Fu and Yang [43]	IVA	Category 1	Interactive, local	Identification rules + suggesting rule	Not mentioned
Zhang and Li et al. [47]	IFSs	Category 2	Interactive, local	Not given	Static
Singh et al. [45]	FLV	Category 2	Interactive, local	Re-evaluation rules	Not mentioned
Xu and Du et al. [48]	numbers	Category 2	Interactive, local	Determination of comprehensive adjustment coefficient + identification of minority opinions and non-cooperative behaviors	Dynamic

ULI: unbalanced linguistic information. MULI: multigranular uncertain linguistic information. IFNs: intuitionistic fuzzy numbers. MLT: multigranular linguistic terms. IVA: interval-valued assessments. IFSs: intuitionistic fuzzy sets. FLV: fuzzy linguistic variables. ULV: uncertain linguistic variable. Category 1 means consensus measures based on the degree of similarity between experts. Category 2 means consensus measures based on the degree of similarity to the collective opinion.

In a word, the aim of this paper is to develop an adaptive method for MAGDM with uncertain linguistic information to obtain the final consensus decision efficiently. The rest of the paper is organized as follows. Basic concepts, operational laws and comparison methods of uncertain linguistic variables are briefly reviewed in Section 2. Section 3 firstly defines the dominance index to compare and rank the uncertain linguistic variables and discusses some properties and advantages of the dominance index. Section 4 defines closeness to compute the similar degree between two experts, based on which the consensus indices at different levels are proposed. Section 5 establishes an adaptive consensus support model for MAGDM problems under uncertain linguistic environment. Section 6 illustrates the practicality and effectiveness of the proposed method through deeply analyzing a typical example of MAGDM problem. Section 7 compares and analyzes the similarities and differences between the proposed method and the other four MAGDM methods under uncertain linguistic environment in detail. Section 8 discusses some comparisons with previous papers and concludes this paper with future perspectives.

**2. Preliminaries**

Firstly, basic concepts, operational laws and common comparison methods related to uncertain linguistic variables are briefly described in the following.

**2.1. Basic concepts and operational laws**

Suppose that  $S = \{s_\alpha | \alpha = 0, 1, \dots, l\}$  is the linguistic term set, where  $s_\alpha$  represents a possible value for a linguistic variable, called linguistic term.  $S$  must have the following characteristics [60–62]:

- (1) The set is ordered:  $s_\alpha \geq s_\beta$  if  $\alpha \geq \beta$ ;
- (2) There is the negation operator:  $neg(s_\alpha) = s_\beta$  such that  $\beta = l - \alpha$ ;
- (3) Max operator:  $\max(s_\alpha, s_\beta) = s_\alpha$  if  $s_\alpha \geq s_\beta$ ;
- (4) Min operator:  $\min(s_\alpha, s_\beta) = s_\alpha$  if  $s_\alpha \leq s_\beta$ .

For example, a set of seven terms  $S$  could be

$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$ .

In the integrating process of decision information, the integration results often do not match the elements in  $S$ . In order to facilitate computation and avoid the loss of information, the discrete term set  $S$  can be extended to continuous term set  $\tilde{S} = \{s_\alpha | \alpha \in [0, q]\}$  [54], where  $q$  ( $q \geq l$ ) is a sufficiently large positive integer. The elements included in  $\tilde{S}$  also meet the above corresponding characteristics respectively. For arbitrary linguistic term  $s_\alpha$ , if  $s_\alpha \in S$ , we call  $s_\alpha$  the original term and  $\alpha$  the original term index; otherwise, we call  $s_\alpha$  the virtual term and  $\alpha$  the virtual term index. In general, experts use the original linguistic terms to evaluate alternatives, and the virtual linguistic terms only appear during the operation. Let  $I(s_\alpha)$  denote the term index of  $s_\alpha$  in  $S$ , i.e.,  $I(s_\alpha) = \alpha$ .

In many real-world problems, the input linguistic information may not match any of the original linguistic labels, and they may be located between two of them. In such cases, Xu defined the uncertain linguistic variables and introduced related operational laws [54–56,63].

**Definition 1.** Let  $\tilde{s} = [s_L, s_R]$ , where  $s_L, s_R \in S$ ,  $s_L$  and  $s_R$  are the lower and the upper limits, respectively, we call  $\tilde{s}$  the uncertain linguistic variable.

Let  $\tilde{S}$  be the set of all uncertain linguistic variables. Considering any three uncertain linguistic variables  $\tilde{s} = [s_L, s_R]$ ,  $\tilde{s}_1 = [s_{1L}, s_{1R}]$ , and  $\tilde{s}_2 = [s_{2L}, s_{2R}]$ ,  $\tilde{s}, \tilde{s}_1, \tilde{s}_2 \in \tilde{S}$ ,  $\lambda \in [0, 1]$ , the operational laws could be defined as:

- (1)  $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{1L}, s_{1R}] \oplus [s_{2L}, s_{2R}] = [s_{1L} \oplus s_{2L}, s_{1R} \oplus s_{2R}] = [s_{1L+2L}, s_{1R+2R}]$ ;
- (2)  $\lambda \tilde{s} = \lambda [s_L, s_R] = [\lambda s_L, \lambda s_R] = [s_{\lambda L}, s_{\lambda R}]$ .

**Definition 2.** Let  $\tilde{S} = \{\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n\}$  be a set of the uncertain linguistic variables, where  $\tilde{s}_i = [s_{iL}, s_{iR}] \in \tilde{S}$ . Suppose  $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}$  is the associated weights where  $\omega_i \geq 0, \sum_{i=1}^n \omega_i = 1$ . The uncertain linguistic weighted average (ULWA) operator is defined as

$$ULWA(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) = \omega_1 \tilde{s}_1 \oplus \omega_2 \tilde{s}_2 \oplus \dots \oplus \omega_n \tilde{s}_n \tag{1}$$

**2.2. Comparison methods between uncertain linguistic variables**

Since the uncertain linguistic variables are interval values instead of crisp numbers, how to compare the size of two uncertain linguistic variables is also a problem worthy of study.

Possibility degree is a common method to compare the uncertain linguistic variables [41,54–59], which is given as follows.

**Definition 3.** Let  $\tilde{s}_1 = [s_{1L}, s_{1R}]$  and  $\tilde{s}_2 = [s_{2L}, s_{2R}]$  be two uncertain linguistic variables, and let  $len(\tilde{s}_1) = I(s_{1R}) - I(s_{1L})$ ,  $len(\tilde{s}_2) = I(s_{2R}) - I(s_{2L})$ , then the degree of possibility of  $\tilde{s}_1 \geq \tilde{s}_2$  is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max(0, len(\tilde{s}_1) + len(\tilde{s}_2) - \max(I(s_{2R}) - I(s_{1L}), 0))}{len(\tilde{s}_1) + len(\tilde{s}_2)}. \quad (2)$$

Especially, if both the linguistic variables  $\tilde{s}_1$  and  $\tilde{s}_2$  express precise information (i.e. if  $len(\tilde{s}_1) + len(\tilde{s}_2) = 0$ ), then the degree of possibility of  $\tilde{s}_1 \geq \tilde{s}_2$  is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \begin{cases} 1 & \text{if } \tilde{s}_1 > \tilde{s}_2, \\ 1/2 & \text{if } \tilde{s}_1 = \tilde{s}_2, \\ 0 & \text{if } \tilde{s}_1 < \tilde{s}_2. \end{cases}$$

The possibility degree  $p(\tilde{s}_1 \geq \tilde{s}_2)$  satisfies the following properties:

- (1)  $0 \leq p(\tilde{s}_1 \geq \tilde{s}_2) \leq 1, 0 \leq p(\tilde{s}_2 \geq \tilde{s}_1) \leq 1$ ;
- (2)  $p(\tilde{s}_1 \geq \tilde{s}_2) + p(\tilde{s}_2 \geq \tilde{s}_1) = 1$ . Especially,  $p(\tilde{s}_1 \geq \tilde{s}_1) = p(\tilde{s}_2 \geq \tilde{s}_2) = 1/2$ .

Based on the possibility degree, the procedure for the ranking of the uncertain linguistic variables  $\tilde{s}_i (i = 1, 2, \dots, n)$  is given as follows.

First, compare each  $\tilde{s}_i$  with all the  $\tilde{s}_j (i = 1, 2, \dots, n)$  by using Eq. (1), for simplicity, let  $p_{ij} = p(\tilde{s}_i \geq \tilde{s}_j)$ , then, a complementary matrix  $P = (p_{ij})_{n \times n}$  can be constructed as

$$P = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \dots & p_{nn} \end{bmatrix},$$

where  $p_{ij} \geq 0, p_{ij} + p_{ji} = 1, p_{ii} = 1/2, i, j = 1, 2, \dots, n$ .

Summing all elements in each line of the matrix  $P$ , we have  $p_i = \sum_{j=1}^n p_{ij} (i = 1, 2, \dots, n)$ . Then we can rank the  $\tilde{s}_i (i = 1, 2, \dots, n)$  in descending order in accordance with the values of  $p_i (i = 1, 2, \dots, n)$ .

The superiority index proposed by Xu and Wu et al. [52] is another method to compare two uncertain linguistic variables which is introduced as follows.

The superiority index between two alternatives is defined based on the order relation between uncertain linguistic variables  $\tilde{a}_{ij}$  and  $\tilde{a}_{kj}$  listed in the following, where  $\tilde{a}_{ij}$  represents the performance of alternative  $x_i$  on attribute  $u_j$ ,  $\tilde{a}_{kj}$  represents the performance of alternative  $x_k$  on attribute  $u_j$ .

- $\tilde{a}_{ij} \leq \tilde{a}_{kj}$  if and only if  $a_{ijL} \leq a_{kjL}$  and  $a_{ijR} \leq a_{kjR}$ ;
- $\tilde{a}_{ij} = \tilde{a}_{kj}$  if and only if  $a_{ijL} = a_{kjL}$  and  $a_{ijR} = a_{kjR}$ ;
- $\tilde{a}_{ij} < \tilde{a}_{kj}$  if and only if  $\tilde{a}_{ij} \leq \tilde{a}_{kj}$  and  $\tilde{a}_{ij} \neq \tilde{a}_{kj}$ .

**Definition 4.** For every two alternatives  $x_i, x_k$  and an attribute  $u_j$ , denote

$$b_{ik,j} = \begin{cases} 1, & \text{if } \tilde{a}_{kj} < \tilde{a}_{ij}, \\ 1/2, & \text{if } \tilde{a}_{ij} = \tilde{a}_{kj}, \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

The superiority indices have the following properties:

- (1)  $b_{ii,j} = \frac{1}{2}, j \in (1, \dots, m)$  and  $b_{ii} = \frac{1}{2}m$ ;
- (2)  $0 \leq b_{ik} \leq m$ ;
- (3)  $\frac{1}{2}m \leq S_i \leq m(n - 1) + \frac{1}{2}m$ .

In accordance with the value of  $S_i$ , all the alternatives can be ranked and the best one(s) can be selected. If there are two alternatives that have the same index, the order can be determined by

extracting the original information of two such alternatives in the decision matrix and computing their superiority indices separately.

### 3. Uncertain linguistic dominance index

In this section, for comparing and ranking the uncertain linguistic variables simply and efficiently the concept of dominance index is proposed, which can well reflect the characteristics of the uncertain linguistic information and are more suitable for comparing uncertain linguistic variables.

#### 3.1. Basic concept and properties

As we know that boundedness and orderliness are two salient characteristics of linguistic term set. For a given linguistic term set  $S = \{s_\alpha | \alpha = 0, 1, \dots, l\}$ , its lower and upper limits are  $s_0$  and  $s_l$  respectively. Suppose  $\tilde{s} = [s_L, s_R]$  is an uncertain linguistic variable, then the farther  $I(s_L)$  and  $I(s_R)$  are from the lower limit  $I(s_0)$ , the more dominant  $\tilde{s}$  is; the closer  $I(s_L)$  and  $I(s_R)$  are to the upper limit  $I(s_l)$ , the more dominant  $\tilde{s}$  is. Combined with the above characteristics, the dominance index ( $DI$ ) is given as follows.

**Definition 5.** Let  $\tilde{s}_i = [s_{iL}, s_{iR}]$  be an uncertain linguistic variable, where  $\tilde{s}_i \in \tilde{S}$ , i.e.,  $s_{iL}, s_{iR} \in S$ , then the dominance index of  $\tilde{s}_i$  is defined as

$$DI(\tilde{s}_i) = \frac{\sqrt{[I(s_{iL})]^2 + [I(s_{iR})]^2}}{\sqrt{[I(s_{iL}) - l]^2 + [I(s_{iR}) - l]^2} + \sqrt{[I(s_{iL})]^2 + [I(s_{iR})]^2}}. \quad (4)$$

Especially, if  $I(s_{iL}) = I(s_{iR}) = 0$ , then the dominance index of  $\tilde{s}_i$  is 0, i.e.,  $DI(s_0) = 0$ ; if  $I(s_{iL}) = I(s_{iR}) = l$ , then the dominance index of  $\tilde{s}_i$  is 1, i.e.,  $DI(s_l) = 1$ .

Obviously,  $DI(\tilde{s}_i)$  has the following properties.

**Property 1.**  $0 \leq DI(\tilde{s}_i) \leq 1$ .

**Property 2.** Let  $\tilde{s}_i, \tilde{s}_h \in \tilde{S}$  be two uncertain linguistic variables, then

- if  $\tilde{s}_i = \tilde{s}_h$ , then  $DI(\tilde{s}_i) = DI(\tilde{s}_h)$ ;
- $\tilde{s}_i > \tilde{s}_h$ , iff  $DI(\tilde{s}_i) > DI(\tilde{s}_h)$ ;
- $\tilde{s}_i < \tilde{s}_h$ , iff  $DI(\tilde{s}_i) < DI(\tilde{s}_h)$ .

Based on Eq. (4), it is not difficult to prove that if  $I(s_{iL}) + I(s_{iR}) = l$ , we have  $DI(\tilde{s}_i) = 0.5$ . Though for two different uncertain linguistic variables,  $\tilde{s}_i$  and  $\tilde{s}_h$ , if  $DI(\tilde{s}_i) = DI(\tilde{s}_h) = 0.5$ , we cannot judge which one is dominant. But fortunately, such situation does not occur frequently, in most cases we are able to take advantage of the dominance index to complete the comparison and ranking of the uncertain linguistic variables.

#### 3.2. Comparative analysis

In the following, we will illustrate the advantages of the dominance index compared with the possibility degree and the superiority index in detail by two concrete examples.

**Example 1.** Let  $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ ,  $\tilde{s}_1 = [s_5, s_6]$ ,  $\tilde{s}_2 = [s_3, s_4]$ ,  $\tilde{s}_3 = [s_1, s_2]$ . In order to rank the above three values, we can use three methods. (1) By using possibility degree we have to complete the following comparisons based on Eq. (2), i.e.,  $p(\tilde{s}_1 \geq \tilde{s}_1) = 0.5$ ,  $p(\tilde{s}_1 \geq \tilde{s}_2) = 1, p(\tilde{s}_1 \geq \tilde{s}_3) = 1, p(\tilde{s}_2 \geq \tilde{s}_1) = 0, p(\tilde{s}_2 \geq \tilde{s}_2) = 0.5, p(\tilde{s}_2 \geq \tilde{s}_3) = 1, p(\tilde{s}_3 \geq \tilde{s}_1) = 0, p(\tilde{s}_3 \geq \tilde{s}_2) = 0, p(\tilde{s}_3 \geq \tilde{s}_3) = 0.5$ . Then, we can get  $p_1 = 2.5, p_2 = 1.5, p_3 = 0.5$ , and the ranking  $\tilde{s}_1 > \tilde{s}_2 > \tilde{s}_3$ ; (2) By using the superiority index we have to complete the following comparisons based on Eq. (3), i.e.,  $b_{11} = 0.5, b_{12} = 1, b_{13} = 1, b_{21} = 0, b_{22} = 0.5, b_{23} = 1, b_{31} = 0, b_{32} = 0, b_{33} = 0.5$ . Then, we can get  $S_1 = 2.5, S_2 = 1.5, S_3 = 0.5$ , and the ranking  $\tilde{s}_1 > \tilde{s}_2 > \tilde{s}_3$ ; (3) By using Eq. (4), we can easily and directly get the same ranking  $\tilde{s}_1 > \tilde{s}_2 > \tilde{s}_3$  based



on the following results of dominance indices:  $DI(\tilde{s}_1) = 0.886$ ,  $DI(\tilde{s}_2) = 0.581$ ,  $DI(\tilde{s}_3) = 0.259$ .

As can be seen from Example 1, the dominance index simplifies the calculation process and is more intuitive and efficient than the other two methods.

**Example 2.** Let  $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ ,  $\tilde{s}_1 = [s_1, s_6]$ ,  $\tilde{s}_2 = [s_2, s_5]$ ,  $\tilde{s}_3 = [s_3, s_4]$ . By using Eq. (2), we get  $p(\tilde{s}_1 \geq \tilde{s}_2) = 0.5$ ,  $p(\tilde{s}_1 \geq \tilde{s}_3) = 0.5$ ,  $p(\tilde{s}_2 \geq \tilde{s}_3) = 0.5$ , i.e., the three uncertain linguistic variables are indistinguishable based on the possibility degree; by using Eq. (3), we find that the comparisons between different uncertain linguistic variables cannot be performed, that is, the superiority index is failed; while, by using Eq. (4), we have  $DI(\tilde{s}_1) = 0.549$ ,  $DI(\tilde{s}_2) = 0.566$ ,  $DI(\tilde{s}_3) = 0.581$ , based on which we can easily get the ranking  $\tilde{s}_3 > \tilde{s}_2 > \tilde{s}_1$ .

From Example 2, we can draw the following conclusions.

- (1) For two different uncertain linguistic variables,  $\tilde{s}_i$  and  $\tilde{s}_h$ , if  $I(s_{iL}) + I(s_{iR}) = I(s_{hL}) + I(s_{hR})$ , then based on the possibility degree the two uncertain linguistic variables are indistinguishable;
- (2) If the two different uncertain linguistic variables do not satisfy the order relation listed in Definition 4, they cannot be compared based on the superiority index.
- (3) The ranking result can be easily obtained by using the dominance index.

Through the comparative analysis we can see that the dominance index is more suitable for comparing the uncertain linguistic information in most cases. The specific performance of the dominance index is as follows: (1) The calculation process of the dominance index is simpler and more direct; (2) The measurement results of the dominance index are more intuitive and easier to compare and rank.

#### 4. Multi-level consensus indices for uncertain linguistic MAGDM

Consider a MAGDM problem with uncertain linguistic information. Let  $X = \{x_1, x_2, \dots, x_n\}$  be a discrete set of alternatives,  $U = \{u_1, u_2, \dots, u_m\}$  be the set of attributes. Let  $\omega = (\omega_1, \omega_2, \dots, \omega_m)^T$  be the weight vector of attributes, where  $\omega_j \geq 0, j = 1, 2, \dots, m, \sum_{j=1}^m \omega_j = 1$ . Let  $E = \{e_1, e_2, \dots, e_t\}$  be the set of experts, and  $v = (v_1, v_2, \dots, v_t)^T$  be the weight vector of experts.  $\tilde{S}$  is the set of all uncertain linguistic variables.  $\tilde{A} = \{\tilde{A}^{(1)}, \tilde{A}^{(2)}, \dots, \tilde{A}^{(t)}\}$  is the set of decision matrices of  $t$  experts, where  $\tilde{A}^{(k)} = (\tilde{s}_{ij}^{(k)})_{n \times m}$  is the uncertain linguistic decision matrix, and  $\tilde{s}_{ij}^{(k)} = [s_{ijL}^k, s_{ijR}^k] \in \tilde{S}$  is a preference value in the form of uncertain linguistic variable, given by expert  $e_k \in E$  for alternative  $x_i \in X$  with respect to attribute  $u_j \in U (1 \leq k \leq t, 1 \leq i \leq n, 1 \leq j \leq m)$ .

In this section, in order to obtain more accurate feedback and guidance information we will establish multi-level consensus indices to measure and analyze the consensus degree among the experts' opinions at different levels.

First of all, the closeness between two experts is introduced.

**Definition 6.** The closeness of lower limits for alternative  $x_i$  with respect to attribute  $u_j$  between experts  $e_k$  and  $e_h$  is defined as

$$UC_{ijL}(e_k, e_h) = \frac{1 - |I(s_{ijL}^k) - I(s_{ijL}^h)|}{1} \quad (5)$$

**Definition 7.** The closeness of upper limits for alternative  $x_i$  with respect to attribute  $u_j$  between experts  $e_k$  and  $e_h$  is defined as

$$UC_{ijR}(e_k, e_h) = \frac{1 - |I(s_{ijR}^k) - I(s_{ijR}^h)|}{1} \quad (6)$$

**Definition 8.** The closeness for alternative  $x_i$  with respect to attribute  $u_j$  between experts  $e_k$  and  $e_h$  is defined as

$$UC_{ij}(e_k, e_h) = \frac{1}{2}(UC_{ijL}(e_k, e_h) + UC_{ijR}(e_k, e_h)) \quad (7)$$

**Definition 9.** The total closeness between experts  $e_k$  and  $e_h$  is defined as

$$UC(e_k, e_h) = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^m \omega_j UC_{ij}(e_k, e_h) \quad (8)$$

obviously, the above definitions of closeness have the following properties:

- (1) (Boundedness)  $0 \leq UC_{ijL}(e_k, e_h) \leq 1; 0 \leq UC_{ijR}(e_k, e_h) \leq 1; 0 \leq UC_{ij}(e_k, e_h) \leq 1; 0 \leq UC(e_k, e_h) \leq 1;$
- (2) (Symmetry)  $UC_{ijL}(e_k, e_h) = UC_{ijL}(e_h, e_k); UC_{ijR}(e_k, e_h) = UC_{ijR}(e_h, e_k); UC_{ij}(e_k, e_h) = UC_{ij}(e_h, e_k); UC(e_k, e_h) = UC(e_h, e_k).$

In the following, based on the closeness, the consensus indices at different levels are defined.

**Definition 10.** The consensus index on lower limits for alternative  $x_i$  with respect to attribute  $u_j$  between expert  $e_k$  and the group is defined as

$$GCI_{ijL}(e_k) = \sum_{h=1}^t v_h UC_{ijL}(e_k, e_h) \quad (9)$$

**Definition 11.** The consensus index on upper limits for alternative  $x_i$  with respect to attribute  $u_j$  between expert  $e_k$  and the group is defined as

$$GCI_{ijR}(e_k) = \sum_{h=1}^t v_h UC_{ijR}(e_k, e_h) \quad (10)$$

**Definition 12.** The consensus index on lower limits for alternative  $x_i$  between expert  $e_k$  and the group is defined as

$$GCI_{iL}(e_k) = \sum_{j=1}^m \omega_j GCI_{ijL}(e_k) \quad (11)$$

**Definition 13.** The consensus index on upper limits for alternative  $x_i$  between expert  $e_k$  and the group is defined as

$$GCI_{iR}(e_k) = \sum_{j=1}^m \omega_j GCI_{ijR}(e_k) \quad (12)$$

**Definition 14.** The consensus index for alternative  $x_i$  between expert  $e_k$  and the group is defined as

$$GCI_{xi}(e_k) = \frac{1}{2}(GCI_{iL}(e_k) + GCI_{iR}(e_k)) \quad (13)$$

**Definition 15.** The consensus index between expert  $e_k$  and the group is defined as

$$GCI(e_k) = \frac{1}{n} \sum_{i=1}^n GCI_{xi}(e_k) = \sum_{h=1}^t v_h UC(e_k, e_h) \quad (14)$$

**Definition 16.** The group consensus index is defined as

$$GCI = \sum_{k=1}^t v_k GCI(e_k) \quad (15)$$

It is worth emphasizing that the consensus indices defined above take into account the weights of experts and attributes, which make the above consensus measures more reasonable. Obviously,  $0 \leq GCI(e_k) \leq 1, 0 \leq GCI \leq 1$ . The larger the value of  $GCI(e_k)$ ,

the closer that the expert  $e_k$  is to the group. If  $GCI(e_k)=1$ , then the expert  $e_k$  is in full consensus with the group opinions. The threshold of consensus level  $\gamma$  can be determined by the experts in advance according to the actual situation. When for each expert  $e_k \in E$ ,  $GCI(e_k) \geq \gamma$ , it can be concluded that an acceptable level of consensus is achieved among the experts.

**5. Adaptive consensus method based on multi-level consensus indices and dominance indices**

In this section, an adaptive consensus method for MAGDM under uncertain linguistic environment is established, which can adjust the experts' weights adaptively and help the experts reach an acceptable level of the group consensus. The main ideas of the method are described as follows. (1) Based on the closeness and consensus indices, a non-linear programming model is constructed to determine the current optimal experts' weights by maximizing the group consensus; (2) By calculating the optimized multi-level consensus indices, the assessment values that need to be adjusted can be accurately located; (3) Based on the current optimal experts' weights, the reference values corresponding to the assessment values that need to be adjusted are given to guide the experts to modify their opinions selectively and effectively which reduce the blindness of the adjustment; (4) By using the dominance indices, the final ranking of all the alternatives can be obtained more conveniently and the best one(s) can be selected.

**5.1. Adaptive experts' weights**

In the consensus reaching process, when experts adjust their individual opinions, the experts' weights should also be changed accordingly. In order to obtain the more reasonable experts' weights under the premise of maximizing the group consensus, based on Eqs. (14) and (15) a non-linear programming model is constructed to dynamically adjust the experts' weights.

$$\begin{aligned} \max GCI^{(\ell)*} &= \sum_{k=1}^t v_k^{(\ell)*} \sum_{h=1}^t v_h^{(\ell)*} UC^{(\ell)}(e_k, e_h) \\ \text{s.t.} &\begin{cases} \sum_{h=1}^t v_h^{(\ell)*} UC^{(\ell)}(e_k, e_h) \geq GCI^{(\ell)}(e_k), k = 1, \dots, t \\ \sum_{k=1}^t v_k^{(\ell)*} = 1 \\ v_k^{(\ell)*} > 0, k = 1, 2, \dots, t \end{cases} \quad (M-1) \end{aligned}$$

where  $\ell$  is the practical number of rounds whose original value is 0 and the experts' initial weights are equal. The constraint condition  $\sum_{h=1}^t v_h^{(\ell)*} UC^{(\ell)}(e_k, e_h) \geq GCI^{(\ell)}(e_k)$ , ( $k = 1, \dots, t$ ) ensures that the optimized consensus degree between each expert and the group is not less than the previous consensus degree.

By solving the above model using the Lingo software package, the current optimal experts' weights  $v_k^{(\ell)*}(k=1, 2, \dots, t)$  can be obtained. It is worth noting that  $v_k^{(\ell)*}(k=1, 2, \dots, t)$  are only the optimal weights of the  $\ell$ th round. Since the consensus reaching process is a dynamic iterative process, when the individual opinions are adjusted by the experts, the experts' weights obtained by the model (M-1) will also be changed accordingly. So the model (M-1) has local optimum rather than global optimum.

**5.2. The feedback mechanism**

In order to retain the experts' original assessment values as much as possible, adjust the assessment values as little as possible, and point out the direction and degree of the adjustments as accu-

rately as possible, identification rules and recommendation rule are formulated as follows.

**Identification rules (IRs)** are given to locate the assessment values that need to be adjusted. In order to determine which expert need to modify his/her opinions and in which specific position he/she need to modify, the identification rules are designed at three different levels based on multi-level consensus indices, i.e., the experts, the alternatives and the lower limits or upper limits of the assessment values that need to be changed.

- (1) **Identification rule for the experts (IR1):** Determine the experts who need to reevaluate their individual opinions. If  $GCI^{(\ell)*}(e_{k'}) < \gamma$  (where  $e_{k'} \in E$ ), the decision matrix  $\tilde{A}^{(k')(\ell)} = (\tilde{s}_{ij}^{k'(\ell)})_{n \times m}$  needs to be adjusted by the expert  $e_{k'}$ . We denote all the experts whose consensus indices do not meet the pre-defined consensus level by the set  $E'$ . For each expert  $e_{k'} \in E'$ , calculate the consensus indices at different levels  $GCI_{ijL}^{(\ell)*}(e_{k'})$ ,  $GCI_{ijR}^{(\ell)*}(e_{k'})$ ,  $GCI_{iL}^{(\ell)*}(e_{k'})$ ,  $GCI_{iR}^{(\ell)*}(e_{k'})$  and  $GCI_{x_i}^{(\ell)*}(e_{k'})$  ( $i = 1, \dots, n$ ;  $j = 1, \dots, m$ ) by using Eqs. (9)–(13).
- (2) **Identification rule for the alternatives (IR2):** Determine the alternatives that need to be adjusted under each expert in  $E'$ . Suppose  $GCI_{x_{j'}}^{(\ell)*}(e_{k'}) = \min_{1 \leq i \leq n} (GCI_{x_i}^{(\ell)*}(e_{k'}))$ , then the assessment values for alternative  $x_{j'}$  in  $\tilde{A}^{(k')(\ell)}$  need to be adjusted by the expert  $e_{k'}$ . We denote the alternatives that need to be adjusted by  $x_{j'}^{(k')(\ell)}$  and the set of all the alternatives that need to be adjusted by  $X'$ .
- (3) **Identification rule for the lower or upper limits (IR3):** Determine the lower limits or upper limits of the assessment values that need to be adjusted. For  $\forall j'$  ( $1 \leq j' \leq m$ ), If  $GCI_{i'j'L}^{(\ell)*}(e_{k''}) < \min_{x_{i'}^{k'(\ell)} \in X'} (GCI_{i'L}^{(\ell)*}(e_{k'}), GCI_{i'R}^{(\ell)*}(e_{k'}))$  or  $GCI_{i'j'R}^{(\ell)*}(e_{k''}) < \min_{x_{i'}^{k'(\ell)} \in X'} (GCI_{i'L}^{(\ell)*}(e_{k'}), GCI_{i'R}^{(\ell)*}(e_{k'}))$  (where  $x_{i'}^{k''(\ell)} \in X'$ ), then the corresponding assessment value  $s_{i'j'L}^{k''(\ell)}$  or  $s_{i'j'R}^{k''(\ell)}$  needs to be adjusted by the expert  $e_{k''}$ ; if for the expert  $e_{k''} \in E'$ , the assessment value satisfying the above condition does not exist, then the assessment value satisfying the following equation  $GCI_{i'j'L}^{(\ell)*}(e_{k''}) = \min_{1 \leq j \leq m} (GCI_{i'jL}^{(\ell)*}(e_{k''}), GCI_{i'jR}^{(\ell)*}(e_{k''}))$  or  $GCI_{i'j'R}^{(\ell)*}(e_{k''}) = \min_{1 \leq j \leq m} (GCI_{i'jL}^{(\ell)*}(e_{k''}), GCI_{i'jR}^{(\ell)*}(e_{k''}))$  denoted by  $s_{i'j'L}^{k''(\ell)}$  or  $s_{i'j'R}^{k''(\ell)}$  needs to be adjusted by the expert  $e_{k''}$ ; if there are two lower limits of assessment values denoted by  $s_{i'j'L}^{k'(\ell)}$  and  $s_{i'j'L}^{k''(\ell)}$  which are for the same alternative  $x_{i'}$  with respect to the same attribute  $u_{j'}$  but need to be adjusted by two different experts  $e_{k'}$  and  $e_{k''}$ , suppose  $GCI_{i'j'L}^{(\ell)*}(e_{k'}) < GCI_{i'j'L}^{(\ell)*}(e_{k''})$ , then only  $s_{i'j'L}^{k'(\ell)}$  will be returned to the expert  $e_{k'}$  and need to be adjusted. Similarly, for the upper limits with similar situation the similar processing are also done.

In order to further clarify the direction and degree of the adjustments, the recommendation rule is designed to provide explicit modification suggestions for the experts.

**Recommendation rule (RR):** For each assessment value  $s_{i'j'L}^{k'(\ell)}$  or  $s_{i'j'R}^{k'(\ell)}$  that need to be modified, by using the current optimal experts' weights  $v_k^{(\ell)*}(k=1, 2, \dots, t)$  the corresponding reference value can be obtained and denoted as  $a_{i'j'L}^{k'(\ell)}$  or  $a_{i'j'R}^{k'(\ell)}$ , where  $I(a_{i'j'L}^{k'(\ell)}) = \sum_{h=1}^t v_h^{(\ell)*} I(s_{i'j'L}^{h(\ell)})$  or  $I(a_{i'j'R}^{k'(\ell)}) = \sum_{h=1}^t v_h^{(\ell)*} I(s_{i'j'R}^{h(\ell)})$ .

The reference values can not only provide a more accurate reference for the experts to modify their opinions, but also realize the automatic modification of individual opinions under some special

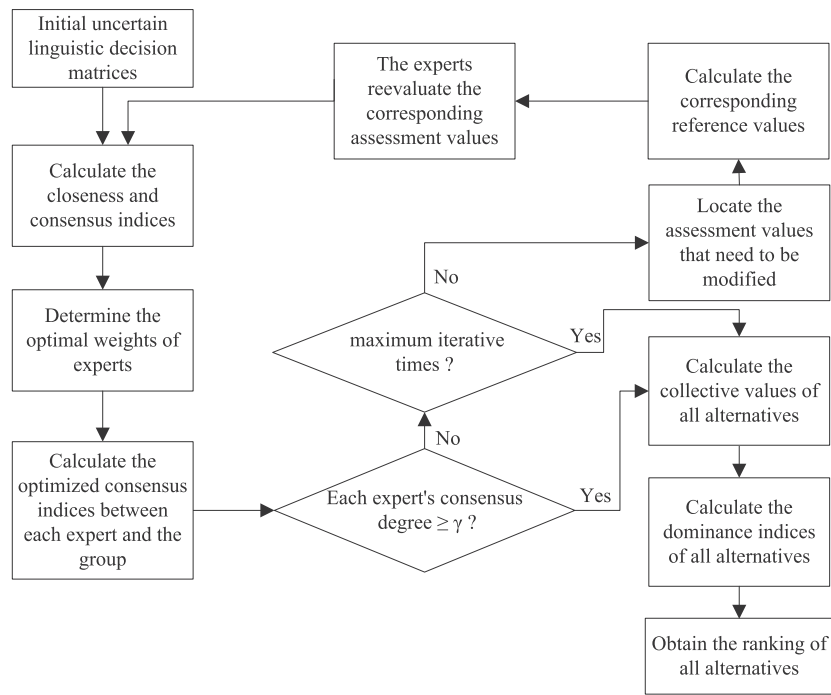


Fig. 1. The schematic illustration of the proposed adaptive consensus process.

conditions, such as the case of unable to obtain the experts' feedback or the situation in which the experts refuse to make changes.

5.3. The procedure of the proposed method

For an uncertain linguistic MAGDM problem, assume that the weights of attributes are known and the experts' initial weights are equal. Assume  $m\ell$  is the maximum number of iterative times,  $\ell$  is the practical number of rounds whose original value is 0,  $\gamma$  is the threshold of consensus level. The steps of the proposed method for MAGDM based on multi-level consensus indices and dominance indices are given as follows:

**Step 1.** Calculate the closeness  $UC_{ijL}^{(\ell)}(e_k, e_h)$ ,  $UC_{ijR}^{(\ell)}(e_k, e_h)$ ,  $UC_{ij}^{(\ell)}(e_k, e_h)$  and  $UC^{(\ell)}(e_k, e_h)$  ( $k, h = 1, \dots, t; i = 1, \dots, n; j = 1, \dots, m$ ) by using Eqs. (5)–(8); Then based on the current experts' weights we can obtain the current consensus indices  $GCI^{(\ell)}(e_k)$  ( $k = 1, \dots, t$ ) by using Eq. (14).

**Step 2.** Determine the current optimal experts' weights  $v_k^{(\ell)*}$  ( $k = 1, 2, \dots, t$ ) by solving the non-linear model (M-1).

**Step 3.** Calculate the optimized consensus indices between each expert and the group by using the optimal experts' weights and Eq. (14), i.e.,  $GCI^{(\ell)*}(e_k) = \sum_{h=1}^t v_h^{(\ell)*} UC^{(\ell)}(e_k, e_h)$  ( $k = 1, \dots, t$ ).

**Step 4.** Compare  $GCI^{(\ell)*}(e_k)$  ( $k = 1, \dots, t$ ) with the threshold  $\gamma$ . If  $GCI^{(\ell)*}(e_k) \geq \gamma$  for  $\forall k$  ( $1 \leq k \leq t$ ) or  $\ell > m\ell$ , then go to Step 7; Else, go to Step 5.

**Step 5.** Locate the assessment values that need to be adjusted by using identification rules.

**Step 6.** Calculate the reference values of the assessment values determined in step 5 by using the recommendation rule and return the assessment values that need to be modified together with the corresponding reference values to the expert  $e_{k'} \in E'$ , at the same time, inform him/her to reevaluate them. Then, collect the updated opinions from the experts.  $\ell = \ell + 1$ ,  $v_k^{(\ell)} = v_k^{(\ell-1)*}$  ( $k = 1, 2, \dots, t$ ). Go to Step 1.

**Step 7.** Based on the updated  $t$  individual decision matrices, by using the known weights of attributes and ULWA operator (see Definition 2) we can obtain the individual overall values of all alter-

natives given by each expert, i.e.,  $\tilde{s}_i^k = \omega_1 \tilde{s}_{i1}^k \oplus \omega_2 \tilde{s}_{i2}^k \oplus \dots \oplus \omega_m \tilde{s}_{im}^k$  ( $k = 1, \dots, t; i = 1, 2, \dots, n$ ); Then by using the final optimal experts' weights and ULWA operator, we can further obtain the collective overall values of all alternatives, i.e.,  $\tilde{s}_i = v_1^{(\ell)*} \tilde{s}_i^1 \oplus v_2^{(\ell)*} \tilde{s}_i^2 \oplus \dots \oplus v_t^{(\ell)*} \tilde{s}_i^t$  ( $i = 1, 2, \dots, n$ ).

**Step 8.** Compute the dominance index of each  $\tilde{s}_i$  denoted by  $DI(\tilde{s}_i)$  ( $i = 1, 2, \dots, n$ ) by using Eq. (4).

**Step 9.** Rank all the alternatives  $x_i$  ( $i = 1, \dots, n$ ) in descending order and select the best one(s) in accordance with the values of  $DI(\tilde{s}_i)$  ( $i = 1, \dots, n$ ).

**Step 10.** End.

In the proposed method, the experts' weights are adjusted adaptively with the changes of experts' opinions and an effective feedback mechanism is used to guide the experts to modify their opinions selectively. Since the feedback mechanism is composed of the identification rules and the recommendation rule, it can reduce the blindness of the adjustment and help the experts reevaluate their individual opinions more effectively. So the proposed method can not only enable experts to find the larger deviation individual opinions automatically and precisely but also provide reference for them to reevaluate their opinions so as to achieve a predefined consensus level.

Fig. 1 illustrates the various steps of the proposed adaptive consensus method.

6. Application example

In this section, the example of risk investment problem is used to illustrate the practicality and effectiveness of the method proposed above. In the example, the four possible alternatives ( $x_1, x_2, x_3, x_4$ ) are to be evaluated by three experts ( $e_1, e_2, e_3$ ) under five attributes:  $u_1$ –the ability of sale,  $u_2$ –the ability of management,  $u_3$ –the ability of production,  $u_4$ –the ability of technology,  $u_5$ –the ability of financing. The linguistic term set used by the experts is  $S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{fair}, s_4 = \text{good}, s_5 = \text{very good}, s_6 = \text{extremely good}\}$ . The weight vector

**Table 2**  
Decision matrix  $\tilde{A}^{(1)}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_0, s_1]$	$[s_2, s_3]$	$[s_3, s_4]$	$[s_3, s_3]$	$[s_2, s_3]$
$x_2$	$[s_0, s_2]$	$[s_2, s_4]$	$[s_2, s_4]$	$[s_2, s_3]$	$[s_2, s_3]$
$x_3$	$[s_0, s_3]$	$[s_2, s_3]$	$[s_2, s_3]$	$[s_0, s_2]$	$[s_3, s_4]$
$x_4$	$[s_0, s_0]$	$[s_3, s_4]$	$[s_1, s_3]$	$[s_3, s_4]$	$[s_0, s_3]$

**Table 3**  
Decision matrix  $\tilde{A}^{(2)}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_1, s_3]$	$[s_1, s_5]$	$[s_4, s_6]$	$[s_3, s_4]$	$[s_4, s_6]$
$x_2$	$[s_2, s_5]$	$[s_3, s_5]$	$[s_3, s_5]$	$[s_4, s_6]$	$[s_3, s_5]$
$x_3$	$[s_2, s_3]$	$[s_3, s_4]$	$[s_4, s_6]$	$[s_0, s_4]$	$[s_4, s_5]$
$x_4$	$[s_0, s_2]$	$[s_5, s_6]$	$[s_2, s_4]$	$[s_4, s_6]$	$[s_1, s_4]$

**Table 4**  
Decision matrix  $\tilde{A}^{(3)}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_1, s_3]$	$[s_3, s_6]$	$[s_5, s_6]$	$[s_5, s_6]$	$[s_3, s_6]$
$x_2$	$[s_1, s_5]$	$[s_5, s_6]$	$[s_3, s_6]$	$[s_2, s_5]$	$[s_3, s_6]$
$x_3$	$[s_1, s_5]$	$[s_5, s_6]$	$[s_5, s_6]$	$[s_1, s_6]$	$[s_6, s_6]$
$x_4$	$[s_0, s_2]$	$[s_6, s_6]$	$[s_2, s_6]$	$[s_3, s_6]$	$[s_2, s_6]$

of attributes is  $\omega = (0.313, 0.121, 0.201, 0.198, 0.167)$ . The decision matrices  $\tilde{A}^{(k)} = (\tilde{s}_{ij}^k)_{n \times m}$  ( $k = 1, 2, 3$ ) are listed in Tables 2–4.

Suppose that the maximum number of iterative times  $m \ell = 6$ , the threshold of consensus level  $\gamma = 0.9$ . The initial value of  $\ell$  is 0. The experts' initial weights are equal, i.e.,  $\nu^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ . Then the whole group decision making process of the example based

**Table 5**  
The closeness  $(UC_{ijL}^{(0)}(e_1, e_2), UC_{ijR}^{(0)}(e_1, e_2))$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	(0.833,0.667)	(0.833,0.667)	(0.833,0.667)	(1.000,0.833)	(0.667,0.500)
$x_2$	(0.667,0.500)	(0.833,0.833)	(0.833,0.833)	(0.667,0.500)	(0.833,0.667)
$x_3$	(0.667,1.000)	(0.833,0.833)	(0.667,0.500)	(1.000,0.667)	(0.833,0.833)
$x_4$	(1.000,0.667)	(0.667,0.667)	(0.833,0.833)	(0.833,0.667)	(0.833,0.833)

**Table 6**  
The closeness  $(UC_{ijL}^{(0)}(e_1, e_3), UC_{ijR}^{(0)}(e_1, e_3))$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	(0.833,0.667)	(0.833,0.500)	(0.667,0.667)	(0.667,0.500)	(0.833,0.500)
$x_2$	(0.833,0.500)	(0.500,0.667)	(0.833,0.667)	(1.000,0.667)	(0.833,0.500)
$x_3$	(0.833,0.667)	(0.500,0.500)	(0.500,0.500)	(0.833,0.333)	(0.500,0.667)
$x_4$	(1.000,0.667)	(0.500,0.667)	(0.833,0.500)	(1.000,0.667)	(0.667,0.500)

**Table 7**  
The closeness  $(UC_{ijL}^{(0)}(e_2, e_3), UC_{ijR}^{(0)}(e_2, e_3))$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	(1.000,1.000)	(0.667,0.833)	(0.833,1.000)	(0.667,0.667)	(0.833,1.000)
$x_2$	(0.833,1.000)	(0.667,0.833)	(1.000,0.833)	(0.667,0.833)	(1.000,0.833)
$x_3$	(0.833,0.667)	(0.667,0.667)	(0.833,1.000)	(0.833,0.667)	(0.667,1.000)
$x_4$	(1.000,1.000)	(0.833,1.000)	(1.000,0.667)	(0.833,1.000)	(0.833,0.667)

**Table 8**  
The closeness  $UC_{ij}^{(0)}(e_1, e_2)$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	0.750	0.750	0.750	0.917	0.583
$x_2$	0.583	0.833	0.833	0.583	0.750
$x_3$	0.833	0.833	0.583	0.833	0.833
$x_4$	0.833	0.667	0.833	0.750	0.833

**Table 9**  
The closeness  $UC_{ij}^{(0)}(e_1, e_3)$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	0.750	0.667	0.667	0.583	0.667
$x_2$	0.667	0.583	0.750	0.833	0.667
$x_3$	0.750	0.500	0.500	0.583	0.583
$x_4$	0.833	0.583	0.667	0.833	0.583

on adaptive experts' weights and effective feedback mechanism is shown as follows:

**Step 1.** Calculate the initial closeness and consensus indices.

- (1) By using Eqs. (5) and (6), The results of closeness  $(UC_{ijL}^{(0)}(e_k, e_h), UC_{ijR}^{(0)}(e_k, e_h))$  are obtained and shown in Tables 5–7.
- (2) By using Eq. (7), The results of closeness  $UC_{ij}^{(0)}(e_k, e_h)$  ( $k, h = 1, 2, 3$  and  $k < h; i = 1, \dots, 4; j = 1, \dots, 5$ ) are obtained and shown in Tables 8–10, based on which we can further obtain  $UC^{(0)}(e_1, e_2) = 0.757, UC^{(0)}(e_1, e_3) = 0.680, UC^{(0)}(e_2, e_3) = 0.856$  by using Eq. (8).
- (3) By using the experts' weights  $\nu^{(0)} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  and Eq. (14), we can obtain the consensus indices  $GCI^{(0)}(e_1) = 0.812, GCI^{(0)}(e_2) = 0.871, GCI^{(0)}(e_3) = 0.845$ .



**Table 10**  
The closeness  $UC_{ij}^{(0)}(e_2, e_3)$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	1.000	0.750	0.917	0.667	0.917
$x_2$	0.917	0.750	0.917	0.750	0.917
$x_3$	0.750	0.667	0.917	0.750	0.833
$x_4$	1.000	0.917	0.833	0.917	0.750

**Step 2.** Construct the following non-linear programming model to get the optimal experts' weights by using model (M-1).

$$\begin{aligned} \max GCI^{(0)*} &= \sum_{k=1}^3 v_k^{(0)*} \sum_{h=1}^3 v_h^{(0)*} UC^{(0)}(e_k, e_h) \\ \text{s.t.} &\begin{cases} \sum_{h=1}^3 v_h^{(0)*} UC^{(0)}(e_k, e_h) \geq GCI^{(0)}(e_k), k = 1, 2, 3 \\ \sum_{k=1}^3 v_k^{(0)*} = 1 \\ v_k^{(0)*} > 0, k = 1, 2, 3 \end{cases} \end{aligned}$$

Based on the results of Step 1, the above model can be expressed in detail as follows:

$$\begin{aligned} \max & v_1^{(0)*}(v_1^{(0)*} + 0.757v_2^{(0)*} + 0.680v_3^{(0)*}) + v_2^{(0)*}(0.757v_1^{(0)*} \\ & + v_2^{(0)*} + 0.856v_3^{(0)*}) + v_3^{(0)*}(0.680v_1^{(0)*} + 0.856v_2^{(0)*} + v_3^{(0)*}) \\ \text{s.t.} &\begin{cases} v_1^{(0)*} + 0.757v_2^{(0)*} + 0.680v_3^{(0)*} \geq 0.812 \\ 0.757v_1^{(0)*} + v_2^{(0)*} + 0.856v_3^{(0)*} \geq 0.871 \\ 0.680v_1^{(0)*} + 0.856v_2^{(0)*} + v_3^{(0)*} \geq 0.845 \\ \sum_{k=1}^3 v_k^{(0)*} = 1 \\ v_1^{(0)*} > 0, v_2^{(0)*} > 0, v_3^{(0)*} > 0 \end{cases} \end{aligned}$$

By solving the above model, the optimal experts' weights are obtained,  $v_1^{(0)*} = 0.330$ ,  $v_2^{(0)*} = 0.343$ ,  $v_3^{(0)*} = 0.327$ .

**Step 3.** Based on the optimal experts' weights, the optimized consensus indices between each expert and the group are obtained by using Eq. (14),  $GCI^{(0)*}(e_1) = 0.812$ ,  $GCI^{(0)*}(e_2) = 0.872$ ,  $GCI^{(0)*}(e_3) = 0.845$ .

**Step 4.** Since  $GCI^{(0)*}(e_1) = 0.812 < 0.9$ ,  $GCI^{(0)*}(e_2) = 0.872 < 0.9$ ,  $GCI^{(0)*}(e_3) = 0.845 < 0.9$ , and  $\ell = 0 < 6$ , then go to Step 5.

**Step 5.** Locate the assessment values that need to be adjusted.

**Table 11**  
The consensus indices ( $GCI_{ijL}^{(0)*}(e_k)$ ,  $GCI_{ijR}^{(0)*}(e_k)$ ).

Experts	Alternatives	Attributes				
		$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$e_1$	$x_1$	(0.888,0.777)	(0.888,0.722)	(0.834,0.777)	(0.891,0.779)	(0.831,0.665)
	$x_2$	(0.831, <b>0.665</b> )	(0.779,0.834)	(0.888,0.834)	(0.886, <b>0.720</b> )	(0.888, <b>0.722</b> )
	$x_3$	(0.831,0.891)	(0.779,0.779)	( <b>0.722,0.665</b> )	(0.946,0.668)	(0.779,0.834)
	$x_4$	(1.000,0.777)	(0.722,0.777)	(0.888,0.779)	(0.943,0.777)	(0.834,0.779)
$e_2$	$x_1$	(0.945,0.890)	(0.836,0.836)	(0.891,0.890)	(0.891,0.836)	(0.836,0.835)
	$x_2$	(0.836,0.835)	(0.836,0.891)	(0.888,0.891)	( <b>0.781,0.781</b> )	(0.945,0.836)
	$x_3$	(0.836,0.891)	(0.836,0.836)	(0.836,0.835)	(0.946,0.781)	(0.836,0.945)
	$x_4$	(1.000,0.890)	(0.836,0.890)	(0.945,0.836)	(0.891,0.890)	(0.891,0.836)
$e_3$	$x_1$	(0.945,0.890)	(0.831,0.778)	(0.833,0.890)	(0.776,0.721)	(0.888,0.835)
	$x_2$	(0.888,0.835)	(0.721,0.833)	(0.945,0.833)	(0.886,0.833)	(0.945,0.778)
	$x_3$	(0.888,0.776)	( <b>0.721,0.721</b> )	(0.778,0.835)	(0.888, <b>0.666</b> )	( <b>0.721,0.890</b> )
	$x_4$	(1.000,0.890)	(0.778,0.890)	(0.945,0.721)	(0.943,0.890)	(0.833,0.721)

The bold values denote the corresponding assessment values that need to be adjusted.

**Table 12**  
The consensus indices ( $GCI_{iL}^{(0)*}(e_k)$ ,  $GCI_{iR}^{(0)*}(e_k)$ ) and  $GCI_{x_i}^{(0)*}(e_k)$ .

Experts	Alternatives	( $GCI_{iL}^{(0)*}(e_k)$ , $GCI_{iR}^{(0)*}(e_k)$ )	$GCI_{x_i}^{(0)*}(e_k)$
$e_1$	$x_1$	(0.868,0.752)	0.810
	$x_2$	(0.857, <b>0.740</b> )	<b>0.798</b>
	$x_3$	(0.817,0.778)	<b>0.798</b>
	$x_4$	(0.905,0.778)	0.841
$e_2$	$x_1$	(0.892,0.864)	0.878
	$x_2$	(0.865,0.842)	<b>0.854</b>
	$x_3$	(0.857,0.860)	0.859
	$x_4$	(0.929,0.870)	0.900
$e_3$	$x_1$	(0.866,0.834)	0.850
	$x_2$	(0.888,0.824)	0.856
	$x_3$	(0.818,0.778)	<b>0.798</b>
	$x_4$	(0.923,0.828)	0.875

The bold values in the last column denote the alternatives whose assessment values need to be adjusted by each expert; the bold value 0.740 represents a threshold obtained by IR3, below which the lower or upper limits of the assessment values need to be adjusted. The corresponding details are given in Step 5 of Section 6.

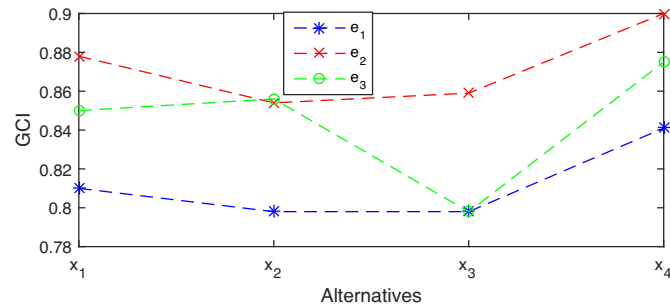
- Determine the experts whose opinions need to be adjusted by using IR1. According to the comparison results of Step 4, the decision matrix provided by each expert needs to be adjusted.
- Based on Tables 5–7, by using the optimal experts' weights and Eqs. (9) and (10), the results of consensus indices ( $GCI_{ijL}^{(0)*}(e_k)$ ,  $GCI_{ijR}^{(0)*}(e_k)$ )( $k = 1, 2, 3$ ) are obtained and shown in Table 11. The results of consensus indices ( $GCI_{iL}^{(0)*}(e_k)$ ,  $GCI_{iR}^{(0)*}(e_k)$ ) and  $GCI_{x_i}^{(0)*}(e_k)$  are obtained by using Eqs. (11)–(13) and shown in Table 12.
- According to Table 12, by using IR2 we can determine that for expert  $e_1$ , the assessment values for  $x_2$  and  $x_3$  need to be adjusted; for expert  $e_2$ , the assessment values for  $x_2$  need to be adjusted; for expert  $e_3$ , the assessment values for  $x_3$  need to be adjusted (see Fig. 2).
- According to Table 11, by using IR3 we can further determine that for expert  $e_1$ , the assessment values  $s_{21R}^{1(0)}$ ,  $s_{24R}^{1(0)}$ ,  $s_{25R}^{1(0)}$ ,  $s_{33L}^{1(0)}$  and  $s_{33R}^{1(0)}$  need to be adjusted; for expert  $e_2$ ,  $s_{24L}^{2(0)}$  needs to be adjusted; for expert  $e_3$ ,  $s_{32L}^{3(0)}$ ,  $s_{32R}^{3(0)}$ ,  $s_{34R}^{3(0)}$  and  $s_{35L}^{3(0)}$  need to be adjusted (see Fig. 3 and Fig. 4).

**Step 6.** Guide the experts to modify the above assessment values by using the recommendation rule.

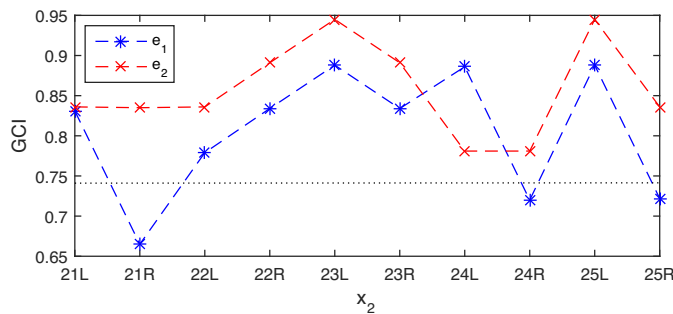
- Calculate the corresponding reference values. By using the optimal experts' weights, we can get the corresponding reference values. They are  $a_{21R}^{1(0)} = s_{4.01}$ ,  $a_{24R}^{1(0)} = s_{4.683}$ ,  $a_{25R}^{1(0)} = s_{4.667}$ ,

**Table 13**  
The consensus indices and optimal experts' weights in each round.

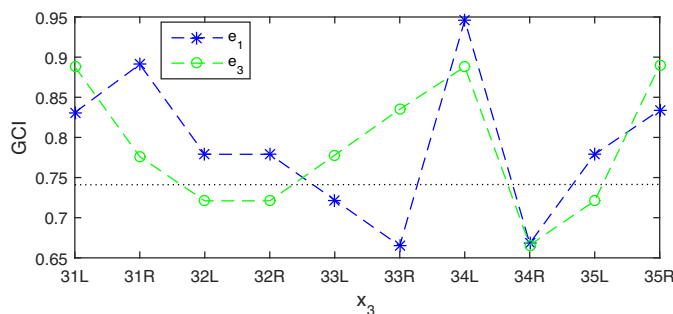
Round	$v_1^{(l)*}$	$v_2^{(l)*}$	$v_3^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	Modified values
1	0.332	0.340	0.328	0.852	0.898	0.878	$\{s_{12R}^{1(1)}, s_{15R}^{1(1)}, s_{14R}^{3(1)}\}$
2	0.328	0.342	0.330	0.862	0.905	0.888	$\{s_{41R}^{1(2)}, s_{42L}^{1(2)}, s_{42R}^{1(2)}, s_{44R}^{1(2)}, s_{43R}^{3(2)}, s_{45R}^{3(2)}\}$
3	0.331	0.342	0.327	0.880	0.916	0.904	$\{s_{11R}^{1(3)}, s_{13L}^{1(3)}, s_{13R}^{1(3)}, s_{15L}^{1(3)}\}$
4	0.331	0.342	0.327	0.892	0.922	0.910	$\{s_{31L}^{1(4)}, s_{34R}^{1(4)}, s_{35R}^{1(4)}\}$
5	0.312	0.357	0.334	0.901	0.927	0.915	



**Fig. 2.** The consensus index for each alternative between each expert and the group.



**Fig. 3.** The consensus index on lower and upper limits for  $x_2$  under each attribute between  $e_1, e_2$  and the group.



**Fig. 4.** The consensus index on lower and upper limits for  $x_3$  under each attribute between  $e_1, e_3$  and the group.

$a_{33L}^{1(0)} = s_{3.667}, a_{33R}^{1(0)} = s_{5.01}, a_{24L}^{2(0)} = s_{3.343}, a_{32L}^{3(0)} = s_{3.324}, a_{32R}^{3(0)} = s_{4.324}, a_{34R}^{3(0)} = s_{3.994}, a_{35L}^{3(0)} = s_{4.324}$  respectively.

(2) The experts reevaluate the corresponding assessment values according to the above feedback information. Assume the new assessment values updated by the experts are listed as follows:  $s_{21R}^{1(0)} = s_4, s_{24R}^{1(0)} = s_5, s_{25R}^{1(0)} = s_5, s_{33L}^{1(0)} = s_4, s_{33R}^{1(0)} = s_5, s_{24L}^{2(0)} = s_3, s_{32L}^{3(0)} = s_3, s_{32R}^{3(0)} = s_4, s_{34R}^{3(0)} = s_4, s_{35L}^{3(0)} = s_4$ .

**Table 14**  
The updated decision matrix  $\tilde{A}^{(1)*}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_0, s_2]$	$[s_2, s_5]$	$[s_4, s_5]$	$[s_3, s_3]$	$[s_3, s_5]$
$x_2$	$[s_0, s_4]$	$[s_2, s_4]$	$[s_2, s_4]$	$[s_2, s_5]$	$[s_2, s_5]$
$x_3$	$[s_1, s_3]$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_0, s_3]$	$[s_3, s_5]$
$x_4$	$[s_0, s_1]$	$[s_5, s_5]$	$[s_1, s_3]$	$[s_3, s_5]$	$[s_0, s_3]$

The bold values denote the changed assessment values after five rounds of adjustments which have been illustrated in Section 6 before Step 7.

**Table 15**  
The updated decision matrix  $\tilde{A}^{(2)*}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_1, s_3]$	$[s_1, s_5]$	$[s_4, s_6]$	$[s_3, s_4]$	$[s_4, s_6]$
$x_2$	$[s_2, s_5]$	$[s_3, s_5]$	$[s_3, s_6]$	$[s_3, s_6]$	$[s_3, s_6]$
$x_3$	$[s_2, s_3]$	$[s_3, s_4]$	$[s_4, s_6]$	$[s_0, s_4]$	$[s_4, s_5]$
$x_4$	$[s_0, s_2]$	$[s_5, s_6]$	$[s_2, s_4]$	$[s_4, s_6]$	$[s_1, s_4]$

The bold values denote the changed assessment values after five rounds of adjustments which have been illustrated in Section 6 before Step 7.

**Table 16**  
The updated decision matrix  $\tilde{A}^{(3)*}$ .

Alternatives	Attributes				
	$u_1$	$u_2$	$u_3$	$u_4$	$u_5$
$x_1$	$[s_1, s_3]$	$[s_3, s_6]$	$[s_5, s_6]$	$[s_5, s_4]$	$[s_3, s_6]$
$x_2$	$[s_1, s_5]$	$[s_5, s_6]$	$[s_3, s_6]$	$[s_2, s_5]$	$[s_3, s_6]$
$x_3$	$[s_1, s_5]$	$[s_3, s_4]$	$[s_5, s_6]$	$[s_1, s_4]$	$[s_4, s_6]$
$x_4$	$[s_0, s_2]$	$[s_6, s_6]$	$[s_2, s_4]$	$[s_3, s_6]$	$[s_2, s_4]$

The bold values denote the changed assessment values after five rounds of adjustments which have been illustrated in Section 6 before Step 7.

(3) Collect the updated opinions.  $\ell=1, v_1^{(1)} = 0.330, v_2^{(1)} = 0.343, v_3^{(1)} = 0.327$ . Continue the next round of the consensus reaching process.

The specific consensus reaching process in each round is omitted here and the corresponding consensus indices and optimal weights for each expert are listed in Table 13.

After five rounds of adjustments, the predefined consensus level is achieved. The final three updated individual decision matrices meeting the predefined consensus threshold are obtained and shown in Tables 14–16 where the changed assessment values are emphasized in bold.

The final experts' weights are  $v_1^{(5)*} = 0.312, v_2^{(5)*} = 0.357, v_3^{(5)*} = 0.334$ . For  $GCI^{(5)*}(e_1) = 0.901 > 0.9, GCI^{(5)*}(e_2) = 0.927 > 0.9, GCI^{(5)*}(e_3) = 0.915 > 0.9$ , and  $\ell=5 < 6$ , i.e., the consensus level has met the predefined requirements. Then go to Step 7.

**Step 7.** Based on the updated three individual decision matrices, by using the final experts' weights and the known weights of attributes, the collective overall values of all alternatives are

**Table 17**  
The ranking results of each round.

Round	$v_1^{(l)*}$	$v_2^{(l)*}$	$v_3^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	Ranking results
0	0.330	0.343	0.327	0.812	0.872	0.845	$x_1 > x_3 > x_2 > x_4$
1	0.332	0.340	0.328	0.852	0.898	0.878	$x_1 > x_3 > x_2 > x_4$
2	0.328	0.342	0.330	0.862	0.905	0.888	$x_1 > x_2 > x_3 > x_4$
3	0.331	0.342	0.327	0.880	0.916	0.904	$x_1 > x_2 > x_3 > x_4$
4	0.331	0.342	0.327	0.892	0.922	0.910	$x_1 > x_2 > x_3 > x_4$
5	0.312	0.357	0.334	0.901	0.927	0.915	$x_2 > x_1 > x_3 > x_4$

obtained as follows:  $\tilde{s}_1 = [s_{2.620}, s_{4.324}]$ ,  $\tilde{s}_2 = [s_{2.196}, s_{5.051}]$ ,  $\tilde{s}_3 = [s_{2.310}, s_{4.371}]$ ,  $\tilde{s}_4 = [s_{1.825}, s_{3.712}]$ .

**Step 8.** By using Eq. (4), The dominance index of each  $\tilde{s}_i$  ( $i = 1, \dots, 4$ ) is computed as follows:  $DI(\tilde{s}_1) = 0.573$ ,  $DI(\tilde{s}_2) = 0.584$ ,  $DI(\tilde{s}_3) = 0.551$ ,  $DI(\tilde{s}_4) = 0.465$ .

**Step 9.** In accordance with the values of  $DI(\tilde{s}_i)$  ( $i = 1, \dots, 4$ ), the ranking of four alternatives is  $x_2 > x_1 > x_3 > x_4$ , which indicates that  $x_2$  is the best alternative.

If we use the possibility degree to compare and rank the above four collective overall values, we can also obtain the same ranking results, but the calculation process is relatively complex. First, by using Eq. (2), we get the following complementary matrix  $P$ :

$$P = \begin{bmatrix} 0.5 & 0.467 & 0.535 & 0.696 \\ 0.533 & 0.5 & 0.558 & 0.68 \\ 0.465 & 0.442 & 0.5 & 0.645 \\ 0.304 & 0.32 & 0.355 & 0.5 \end{bmatrix},$$

Then sum all elements in each line of the matrix  $P$ , we have  $p_1 = 2.198$ ,  $p_2 = 2.271$ ,  $p_3 = 2.052$ ,  $p_4 = 1.479$ . According to the values of  $p_i$  ( $i = 1, \dots, 4$ ), the ranking of four alternatives is  $x_2 > x_1 > x_3 > x_4$ .

If we use the superiority indices to compare and rank the above four collective overall values, we can get the following complementary matrix  $B$  by using Eq. (3),

$$B = \begin{bmatrix} 0.5 & - & - & 1 \\ - & 0.5 & - & 1 \\ - & - & 0.5 & 1 \\ 0 & 0 & 0 & 0.5 \end{bmatrix},$$

where the symbol “-” indicates that the two alternatives cannot be compared.

Then we can only get the calculation results of superiority indices, such as  $S_1 \geq 1.5$ ,  $S_2 \geq 1.5$ ,  $S_3 \geq 1.5$ ,  $S_4 = 0.5$ . So, we can only determine that  $x_4$  is the worst alternative but cannot complete the comparison and ranking of the other three alternatives.

By comparison, we can see that the dominance index is much simpler and more suitable for the comparison and ranking of uncertain linguistic information.

In order to better illustrate the role and impact of the proposed consensus method, we list the ranking results of each round in Table 17. From Table 17, one can see that under different levels of consensus, the ranking results of four alternatives are different. That is, the level of consensus does play an important role in the final decision.

### 7. Comparative analysis

In order to demonstrate the relationship and differences between the proposed method and other MAGDM methods under uncertain linguistic environment and emphasize the advantages and characteristics of the proposed method, in this section the proposed method is used to deal with the specific examples given by [63,56,41,52] respectively and the decision results obtained by the proposed method are compared with those obtained from each literature.

#### (1) Comparison with the MAGDM methods without consensus reaching process [63,56][63,56]

In Xu [63], a problem of evaluating university faculty for tenure and promotion (adapted from [64]) is used to illustrate the validity of the MAGDM method based on uncertain linguistic weighted averaging (ULWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator. In the example, there are five alternatives, three attributes (the weight vector is  $\omega = (0.3, 0.4, 0.3)$ ) and four experts (whose weight vector is  $\nu = (0.24, 0.26, 0.23, 0.27)$ ). A comparison of Xu’s method [63] and the proposed method is shown in Table 18.

As shown in Table 18, the proposed method can get the same ranking result as Xu’s method [63] (It is worth noting that due to the error in the calculation, the ranking result listed in [63] is  $x_3 > x_2 > x_1 > x_4 > x_5$ , but the actual ranking result obtained by Xu’s method [63] should be  $x_3 > x_1 > x_2 > x_5 > x_4$ ). The current optimal weight vector of experts is  $\nu = (0.212, 0.437, 0.178, 0.173)$  and the optimized consensus degree between each expert and the group is  $GCI(e_1) = 0.89$ ,  $GCI(e_2) = 0.934$ ,  $GCI(e_3) = 0.907$ ,  $GCI(e_4) = 0.886$  (see Table 19). If we assume the threshold of consensus level  $\gamma = 0.85$ , then the consensus reaching process could be omitted. If we assume  $\gamma = 0.9$ , after two rounds of adjustments the same ranking result can also be obtained by the proposed method, besides the optimal experts’ weights and the group consensus also change accordingly (see Table 19).

In Xu [56], an investment company example (adapted from [62]) is used to illustrate the validity of the MAGDM method based on induced uncertain linguistic OWA (IULOWA) operators. In the example, there are five alternatives, four attributes (the weight vector is  $\omega = (0.3, 0.4, 0.2, 0.1)$ ) and three experts (whose weight vector is  $\nu = (0.35, 0.25, 0.4)$ ). The linguistic term set used in the example is

$$S = \{s_{-4} = \text{extremely poor}, s_{-3} = \text{very poor}, s_{-2} = \text{poor}, s_{-1} = \text{slightly poor}, s_0 = \text{fair}, s_1 = \text{slightly good}, s_2 = \text{good}, s_3 = \text{very good}, s_4 = \text{extremely good}\}.$$

Here we extended the dominance index (see Definition 5) as follows.

**Definition 17.** For a given linguistic term set  $S = \{s_\alpha | \alpha = -l, \dots, l\}$ , let  $\tilde{s}_i = [s_{iL}, s_{iR}]$  be an uncertain linguistic variable, where  $\tilde{s}_i \in \tilde{S}$ , i.e.,  $s_{iL}, s_{iR} \in S$ , then the dominance index of  $\tilde{s}_i$  is defined as

$$DI(\tilde{s}_i) = \frac{\sqrt{[I(s_{iL}) + l]^2 + [I(s_{iR}) + l]^2}}{\sqrt{[I(s_{iL}) - l]^2 + [I(s_{iR}) - l]^2} + \sqrt{[I(s_{iL}) + l]^2 + [I(s_{iR}) + l]^2}}. \tag{16}$$

A comparison of Xu’s method [56] and the proposed method is shown in Table 20.

As shown in Table 20, the proposed method can get the same ranking result as Xu’s method [56]. In addition, the proposed method can further maximize the group consensus by using the model (M-1) and obtain the optimal experts’ weights  $\nu = (0.343, 0.213, 0.444)$ . For the optimized consensus degree between each expert and the group is very high,  $GCI(e_1) = 0.909$ ,  $GCI(e_2) = 0.926$ ,  $GCI(e_3) = 0.926$ , there is no need for consensus reaching process, thus it is omitted in this example.

By comparison, it is not difficult to find that the proposed method can not only obtain the effective decision results, but also give the accurate consensus degree and the optimal experts’ weights under different consensus degree requirements. That is to say, the proposed method pays more attention to the consensus measurement and consensus reaching process, while Xu’s methods [56,63] mainly focus on the research of aggregation operators.

**Table 18**  
A comparison of Xu's method [63] and the proposed method.

Methods	Aggregation operators	Consensus degree	Consensus reaching process	Experts' weights	Comparison methods	Ranking results
Xu's method [63]	ULWA +ULHA	Not mentioned	Not considered	Static	Possibility degree	$x_3 > x_1 > x_2 > x_5 > x_4$
The proposed method	ULWA	0.9	Being omitted	Adaptive	Dominance index	$x_3 > x_1 > x_2 > x_5 > x_4$

**Table 19**  
The consensus indices and optimal experts' weights of the example in [63] obtained by the proposed method.

Round	$v_1^{(l)*}$	$v_2^{(l)*}$	$v_3^{(l)*}$	$v_4^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	$GCI^{(l)*}(e_4)$	Modified values
0	0.212	0.437	0.178	0.173	0.89	0.934	0.907	0.886	$\{s_{31L}^{1(0)} \rightarrow s_6, s_{31R}^{1(0)} \rightarrow s_7, s_{41L}^{4(0)} \rightarrow s_6, s_{42L}^{4(0)} \rightarrow s_5\}$
1	0.222	0.34	0.208	0.21	0.904	0.919	0.914	0.904	

**Table 20**  
A comparison of Xu's method [56] and the proposed method.

Methods	Aggregation operators	Consensus degree	Consensus reaching process	Experts' weights	Comparison methods	Ranking results
Xu's method [56]	IULOWA + ULWA	Not mentioned	Not considered	Static	Possibility degree	$x_5 > x_2 > x_4 > x_1 > x_3$
The proposed method	ULWA	0.924	Being omitted	Adaptive	Dominance index	$x_5 > x_2 > x_4 > x_1 > x_3$

**(2) Comparison with the MAGDM methods based on interactive consensus reaching process [41][41]**

In Xu [41], a practical MAGDM problem involving the prioritization of a set of information technology improvement projects is used to illustrate the validity of the interactive MAGDM method with multigranular uncertain linguistic information (MULI). In the example, there are six alternatives, three attributes (the weight vector is  $\omega = (0.4, 0.3, 0.3)$ ) and four experts (whose weight vector is  $\nu = (0.3, 0.2, 0.3, 0.2)$ ).

- (i) Experts use the unbalanced linguistic label sets to provide their preference information.

In the example,  $S^{(5)}$  is chosen as the basic unbalanced linguistic label set.

$$S^{(5)} = \{s_{-4}^{(5)}, s_{-2}^{(5)}, s_{-1}^{(5)}, s_{-0.4}^{(5)}, s_0^{(5)}, s_{0.4}^{(5)}, s_1^{(5)}, s_2^{(5)}, s_4^{(5)}\}.$$

Based on the unified uncertain linguistic decision matrices, a comparison of Xu's method [41] and the proposed method in the example using unbalanced linguistic label set is shown in Table 21.

As shown in Table 21, the ranking result obtained by the proposed method is not exactly the same as that obtained by Xu's method [41]. In Xu [41], the predefined consensus degree is 0.9, but the degree of similarity between expert e4 and the group is less than 0.9, so e4 is suggested to reevaluate the elements, especially those with the similarity degrees less than 0.9, including  $s_{13}^4, s_{15}^4, s_{21}^4, s_{26}^4, s_{32}^4, s_{33}^4$  and  $s_{36}^4$  and the ranking result  $x_6 > x_5 > x_2 > x_4 > x_1 > x_3$  is obtained based on the modified matrices. However, when we use the proposed method we can get the following calculation results shown in Table 22. From Table 22, we can see that by using the consensus indices defined by the proposed method all of the initial decision matrices have reached the predefined consensus level without any changes. If we make the same changes to the matrix provided by e4, the proposed method can get the same ranking result as Xu's method [41].

- (ii) Experts use the balanced linguistic label sets to provide their preference information.

In the example,  $\tilde{S}^{(5)}$  is chosen as the basic balanced linguistic label set.

$$\tilde{S}^{(5)} = \{s_{-4}^{(5)}, s_{-3}^{(5)}, s_{-2}^{(5)}, s_{-1}^{(5)}, s_0^{(5)}, s_1^{(5)}, s_2^{(5)}, s_3^{(5)}, s_4^{(5)}\}.$$

Based on the unified uncertain linguistic decision matrices, a comparison of Xu's method [41] and the proposed method in the example using balanced linguistic label set is shown in Table 23.

As shown in Table 23, the ranking result obtained by the proposed method is the same as Xu's method [41]. In Xu [41], the predefined consensus degree is 0.9, but the similarity degree between e1 and the group and the similarity degree between e4 and the group are both less than 0.9, so e1 and e4 are both suggested to reevaluate their elements, respectively, especially those with the low similarity degrees. When we use the proposed method we can get the following calculation results shown in Table 24. From Table 24, we can see that after two rounds of adjustments all decision matrices have reached the predefined consensus level.

Through the comparative analysis of the above two cases, we can get the following conclusions: (1) The formulas used to measure the degree of consensus in Xu's method [41] and the proposed method are different, which leads to that the elements to be modified in the two methods are not the same. (2) In Xu's method [41], the feedback information is not accurate enough and only the experts or their elements that need to be modified are pointed out. However, in the proposed method the values to be modified can be accurate to the upper or lower limits and the corresponding reference values are given to help the experts modify their elements effectively. (3) In Xu's method [41], the experts' weights are static, but in the proposed method the experts' weights are dynamic and adaptive. This helps to improve the group consensus level. (4) In the final comparison and ranking phase, Xu's method [41] uses the degree of possibility to rank two uncertain linguistic values, but the proposed method uses the extended dominance index (see Definition 17) to rank them. The differences between them are discussed in detail in Section 3.2. (5) One disadvantage of the proposed method is that it is not valid for the MAGDM problem with multigranular uncertain linguistic information which however can be solved by Xu's method [41].

**(3) Comparison with the MAGDM methods based on automatic consensus reaching process [52][52]**

In Xu and Wu [52], a MAGDM involving the selection of air-conditioning systems (adapted from [65]) is used to demonstrate the validity of the automatic consensus method for MAGDM under uncertain linguistic setting. In the example, there are five alternatives, four attributes (the weight vector is  $\omega = (0.2188, 0.2257, 0.3580, 0.1976)$ ) and three experts (whose weight vector is  $\nu = (1/3,$



**Table 21**  
A comparison of Xu's method [41] and the proposed method in the example using unbalanced linguistic label set.

Methods	Aggregation operators	Consensus degree	Consensus reaching process	Experts' weights	Comparison methods	Ranking results
Xu's method [41]	ULWA	0.9	Interactive	Static	Possibility degree	$x_6 > x_5 > x_2 > x_4 > x_1 > x_3$
The proposed method	ULWA	0.935	Being omitted	Adaptive	Dominance index	$x_6 > x_5 > x_2 > x_1 > x_4 > x_3$

**Table 22**  
The calculation results of the example using unbalanced linguistic label sets in [41] obtained by the proposed method.

Cases	$v_1^{(l)*}$	$v_2^{(l)*}$	$v_3^{(l)*}$	$v_4^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	$GCI^{(l)*}(e_4)$	Ranking results
Initial matrices	0.278	0.142	0.409	0.171	0.929	0.933	0.946	0.911	$x_6 > x_5 > x_2 > x_1 > x_4 > x_3$
Modified matrices	0.304	0.202	0.293	0.202	0.936	0.939	0.952	0.936	$x_6 > x_5 > x_2 > x_4 > x_1 > x_3$

**Table 23**  
A comparison of Xu's method [41] and the proposed method in the example using balanced linguistic label sets.

Methods	Aggregation operators	Consensus degree	Consensus reaching process	Experts' weights	Comparison methods	Ranking results
Xu's method [41]	ULWA	0.9	Interactive	Static	Possibility degree	$x_5 > x_6 > x_1 > x_2 > x_4 > x_3$
The proposed method	ULWA	0.92	Being omitted	Adaptive	Dominance index	$x_5 > x_6 > x_1 > x_2 > x_4 > x_3$

**Table 24**  
The calculation results of the example using balanced linguistic label sets in [41] obtained by the proposed method.

Round	$v_1^{(l)*}$	$v_2^{(l)*}$	$v_3^{(l)*}$	$v_4^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	$GCI^{(l)*}(e_4)$	Modified values
0	0.3	0.196	0.302	0.202	0.903	0.912	0.924	0.874	$\{s_{31L}^{4(0)} \rightarrow s_{-1.5}, s_{31R}^{4(0)} \rightarrow s_{-0.448}, s_{33R}^{4(0)} \rightarrow s_{0.519}\}$
1	0.293	0.182	0.336	0.189	0.909	0.917	0.930	0.897	$\{s_{41L}^{4(1)} \rightarrow s_{-0.606}, s_{41R}^{4(1)} \rightarrow s_{1.331}\}$
2	0.297	0.195	0.312	0.196	0.912	0.920	0.933	0.908	

**Table 25**  
A comparison of Xu and Wu's method [52] and the proposed method.

Methods	Aggregation operators	Consensus degree	Consensus reaching process	Experts' weights	Comparison methods	Ranking results
Xu and Wu's method [52]	ULWA	0.95	18 iterations	Static	Superiority index	$x_5 > x_4 > x_2 > x_1 > x_3$
The proposed method	ULWA	0.967	5 rounds	Adaptive	Dominance index	$x_5 > x_4 > x_2 > x_1 > x_3$

1/3, 1/3)). A comparison of Xu and Wu's method [52] and the proposed method is shown in Table 25.

As shown in Table 25, the ranking result obtained by the proposed method is the same as Xu and Wu's method [52]. In Xu and Wu's method [52], after 18 iterations the predefined consensus level 0.05 (which is defined based on the deviation measure and can be regarded as complementary to the consensus degree 0.95 defined in the proposed method) is achieved. When we use the proposed method we can get the following calculation results shown in Table 26. From Table 26, we can see that after only five rounds of adjustments all decision matrices have reached the equivalent consensus level.

Through the above comparative analysis, we can get the following conclusions:(1) The formulas used to measure the degree of consensus in Xu and Wu's method [52] and the proposed method are different. The two can be regarded as complementary relationship. (2) In Xu and Wu's method [52], the consensus reaching process is automatic and in each iteration all elements in the decision matrix with maximum deviation will be modified. However, in the proposed method the number of values to be adjusted is relatively small and the values to be adjusted may come from different matrices. In addition, the corresponding reference values are given to help the experts modify their elements effectively. (3) In Xu and Wu's method [52], the experts' weights are static, but in the proposed method the experts' weights are dynamic and adaptive. (4) In the final comparison and ranking phase, Xu and Wu's method [52] uses the superiority index to rank all the alternatives, but the proposed method uses the dominance index (see Definition 5) to rank them. The differences between them are discussed in detail in Section 3.2. (5) One disadvantage of the proposed method is that it is

not valid for the MAGDM problem with unknown attribute weights which however can be solved by Xu and Wu's method [52].

### 8. Discussion and conclusions

In order to reach a high consensus level in MAGDM problems, the two key points are how to evaluate the group consensus accurately and how to guide the experts to adjust their individual opinions effectively. Many researchers have presented various consensus models suitable for different environment based on different consensus measures and feedback mechanisms [41–52]. In this paper, we focused on the consensus problem of MAGDM with uncertain linguistic information and proposed an adaptive method for MAGDM. By defining multi-level consensus indices, the consensus degree at different levels between each expert and the group has been measured. Based on the consensus indices, the experts' weights have been adjusted adaptively with the change of the experts' opinions by constructing a non-linear optimization model. By using the multi-level consensus indices and the optimal experts' weights, a feedback mechanism has been established to automatically locate the assessment values that need to be adjusted and provide corresponding reference values to the experts. In addition, dominance index has been defined to facilitate the comparison and ranking of the uncertain linguistic variables. To illustrate the feasibility and practicality of the proposed method, a risk investment example is demonstrated. Furthermore, a detailed comparison and analysis highlights the advantages and characteristics of this method.

Differences between the proposed method and the existing methods are addressed in the following.

**Table 26**  
The consensus indices and optimal experts' weights of the example in [52] obtained by the proposed method.

Round	$\gamma_1^{(l)*}$	$\gamma_2^{(l)*}$	$\gamma_3^{(l)*}$	$GCI^{(l)*}(e_1)$	$GCI^{(l)*}(e_2)$	$GCI^{(l)*}(e_3)$	Modified values
0	0.328	0.33	0.34	0.898	0.896	0.902	$\{s_{43L}^{1(0)} \rightarrow s_4, s_{43R}^{1(0)} \rightarrow s_5, s_{23L}^{2(0)} \rightarrow s_3, s_{23R}^{2(0)} \rightarrow s_5, s_{24L}^{2(0)} \rightarrow s_3, s_{24R}^{2(0)} \rightarrow s_4, s_{51L}^{3(0)} \rightarrow s_2, s_{51R}^{3(0)} \rightarrow s_4\}$
1	0.33	0.325	0.345	0.92	0.918	0.92	$\{s_{11L}^{1(1)} \rightarrow s_3, s_{12L}^{1(1)} \rightarrow s_3, s_{54R}^{2(1)} \rightarrow s_5, s_{11R}^{3(1)} \rightarrow s_5, s_{14R}^{3(1)} \rightarrow s_4, s_{52R}^{3(1)} \rightarrow s_5, s_{53L}^{3(1)} \rightarrow s_4, s_{53R}^{3(1)} \rightarrow s_5, s_{54L}^{3(1)} \rightarrow s_3\}$
2	0.314	0.32	0.366	0.933	0.93	0.938	$\{s_{23L}^{1(2)} \rightarrow s_4, s_{43L}^{2(2)} \rightarrow s_3, s_{41L}^{3(2)} \rightarrow s_3, s_{41R}^{3(2)} \rightarrow s_4, s_{44R}^{3(2)} \rightarrow s_3\}$
3	0.309	0.316	0.375	0.94	0.937	0.947	$\{s_{31L}^{1(3)} \rightarrow s_1, s_{32R}^{1(3)} \rightarrow s_5, s_{31R}^{2(3)} \rightarrow s_3, s_{34L}^{2(3)} \rightarrow s_3, s_{34R}^{2(3)} \rightarrow s_4, s_{31L}^{3(3)} \rightarrow s_3, s_{33L}^{3(3)} \rightarrow s_1\}$
4	0.317	0.316	0.367	0.948	0.945	0.953	$\{s_{11R}^{1(4)} \rightarrow s_6, s_{12R}^{1(4)} \rightarrow s_4, s_{13L}^{1(4)} \rightarrow s_1, s_{51L}^{2(4)} \rightarrow s_2, s_{53L}^{2(4)} \rightarrow s_4, s_{53R}^{2(4)} \rightarrow s_5, s_{54L}^{2(4)} \rightarrow s_3, s_{54R}^{2(4)} \rightarrow s_4, s_{11L}^{3(4)} \rightarrow s_3, s_{13R}^{3(4)} \rightarrow s_4, s_{14L}^{3(4)} \rightarrow s_3\}$
5	0.297	0.294	0.409	0.965	0.964	0.971	

- (1) Comparison with the automatic methods. The automatic methods can automatically modify the individual opinions and make the group opinions reach consensus [49–52], but for the experts they can not directly participate in the consensus reaching process and give their own subjective views. In addition, most of the automatic consensus methods, such as [49,50,52], need to modify all of the evaluation values given by all experts, which makes the consensus reaching process lack of pertinence and rationality. Compared with the automatic methods, the proposed method is interactive, which can not only take into account the experts' subjective views but also provide the experts with targeted recommendations.
- (2) Comparison with the interactive methods. All the existing interactive methods had feedback mechanisms to ask the concerned experts to modify their evaluations [41,42,44,46,47]. Further, [43] and [45] provided guidance rules for guiding experts to decrease or increase their evaluation. Compared with the existing interactive methods, the proposed method can identify the lower or upper limits of the assessment values that need to be adjusted by using the identification rules and provide the corresponding reference values to the experts by using the recommendation rule. That is, the proposed method can provide the experts not only the direction of adjustment but also the degree of adjustment.
- (3) Comparison of the experts' weights. In most of the existing consensus models, the experts' weights were fixed in the consensus reaching process [41,46,47,49,51,52]. In [44], the experts' weights were variable which could be adjusted by the moderator. In the proposed method, the experts' weights are dynamic and are adjusted adaptively with the change of experts' opinions.
- (4) Comparison with [41,52]. [41] and [52] respectively used the possibility degree and the superiority indices to compare the uncertain linguistic variables. While the proposed method uses dominance indices to compare and rank the uncertain linguistic variables. The differences between [41,52] and the proposed method were discussed in detail in Section 3.2.

In summary, the proposed method enriches decision analysis theory and methodology and has the following characteristics. (1) Multi-level consensus indices can measure the consensus degree at different levels and provide more specific and richer information for in-depth analyzing the group consensus. (2) The adaptive experts' weights can reflect the importance of each expert in the group more precisely and more reasonably. (3) The feedback mechanism combines the advantages of automatic method and interactive method in the consensus reaching process, which can not only reduce the blindness of the adjustment but also fully consider the adjustment

opinions of experts. (4) The dominance index can reflect the characteristics of the uncertain linguistic variables better and simplify the process of comparison and ranking. These results are helpful for further studies on group consensus and decision effect in MAGDM problems.

As mentioned in [52,65], each method has its advantages and disadvantages and none can always perform better than the others in all situations. The proposed method is mainly aimed at the traditional MAGDM problems under uncertain linguistic environment when the group wants to achieve a consensus solution and the number of experts, attributes and alternatives is not too much. Especially compared with some novel consensus models for GDM problems based on preference relations [1,28–34], the proposed method needs to be further enriched and improved. In future works, we will conduct further research in the following aspects. (1) Extend the proposed method to the incomplete linguistic environment; (2) Consider the dynamic MAGDM problems with multiple stages; (3) Manage large group of experts. we are expecting to apply the present work to deal with more practical applications in real life such as personnel examination, supplier selection, engineering management, information retrieval, etc.

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