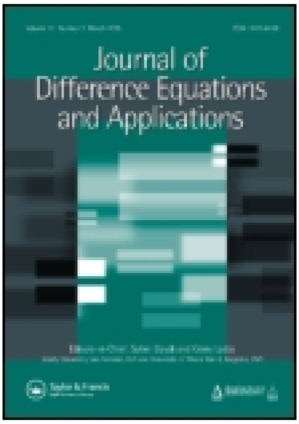

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Existence of periodic positive solutions for a competitive system with two parameters

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Existence of periodic positive solutions for a competitive system with two parameters

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In this paper, a nonlinear competitive system with two parameters is considered. By constructing a curve Δ, we divide the parameters space $\Pi = \{(\alpha, \beta) | (\alpha, \beta) \in R_+^2 \backslash (0,0)\}$ into two disjoint subsets Λ_1 and Λ_2. Furthermore, we obtain the sufficient conditions on the existence of the positive periodic solutions in Λ_1, Λ_2 and on Δ by using the method of upper and lower solutions and the degree theory. More specifically, for the competitive system, there are at least two positive periodic solutions for $(\alpha, \beta) \in \Lambda_1$, at least one positive periodic solution for $(\alpha, \beta) \in \Delta$, and no positive periodic solution for $(\alpha, \beta) \in \Lambda_2$.

Keywords: competitive system; positive periodic solution; fixed point; topological degree

PACS: 87.23.Cc; 45.70.Qj; 89.75.Kd

1. Introduction

In many populations there are recruitment cycles in which the population size at each stage is a function of the population size at the previous stage, and the function is determined by birth and death process. We suppose that the population size changes only through birth and death which occur in the intervals between stages. Denote $x(n)$ as the number of population at stage n, then $x(n+1) - x(n)$ is the number of births minus the number of deaths over the time interval from stage n to stage $n+1$. Suppose further that the birth and death rates are constants b and d, respectively. Then we have

$$x(n+1) - x(n) = (b - d)x(n). \tag{1}$$

In general, equation (1) can be rewritten in the following form:

$$x(n+1) = (1 + b - d)x(n) = rx(n), \quad n = 1, 2 \ldots, \tag{2}$$

where $r = 1 + b - d$ is the intrinsic rate of the population growth.

The case above is the situation with the most insects but is not with many other animals. For example, in some populations, there is a substantial maturation time to sexual

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maturity. In such cases, the delay effect must be included in the model. We can get the model as follows:

$$x(n+1) = r(n)x(n) + b(n)x(n - \tau(n)), \tag{3}$$

where $b(n)$ is a periodic term and $\tau(n)$ is a delay term.

However, there are uncertainty and complexity in the nature world. As a result, we obtain the following model:

$$x(n+1) = a(n)x(n) + h(n)f(n, x(n - \tau(n))), \tag{4}$$

where $a(n)$, $h(n)$ and $\tau(n)$ are T-periodic and T is an integer with $T \geq 1$. $a(n)$, $h(n)$ and $f(x)$ are non-negative with $0 < a(n) < 1$ for all $n \in \{0, 1, \ldots, T-1\}$. This delay difference equation has been studied extensively by numerous scholars in recent years. Some researchers investigated the existence of positive periodic solutions for the nonlinear delay difference equation and almost all of the equations which they studied can be contained in the above form [1,4,10,12,19–21].

Especially, Raffoul [14] and Li et al. [11] considered positive periodic solutions of the difference equations with one parameter:

$$x(n+1) = a(n)x(n) + \lambda h(n)f(x(n - \tau(n))), \tag{5}$$

where $a(n)$, $h(n)$, $\tau(n)$, $f(x)$ are the same as in (4) and λ is non-negative.

Subsequently, Zhang et al. [20] studied a more general equation of the form

$$x(n+1) = a(n)x(n) + h(n)f_1(n, x(n - \tau(n))) - \hat{h}(n)f_2(n, x(n - \tau(n))), \tag{6}$$

where $n \in Z$, $\{h(n)\}_{n\in Z}$ and $\{\widehat{h(n)}\}_{n\in Z}$ are ϖ-positive sequences, $\{\tau(n)\}_{n\in Z}$ is an integer-valued ϖ-periodic sequence, $f_1, f_2 : Z \times R \to R$ are continuous functions, and $f_1(n+\varpi, u) = f_1(n, u), f_2(n+\varpi, u) = f_2(n, u)$ for any $u \in R,\ \ n \in Z$. Based on the fixed-point index theory for a Banach space, positive periodic solutions are found for this system. By using such results, the existence of non-trivial periodic solutions for delay difference equations with positive and negative terms is also considered.

In this paper, in order to well describe the system of two interacting species, we will consider the following system:

$$\begin{cases} x(n+1) = a(n)x(n) + \alpha h(n)f(x(n - \tau_1(n)), \quad y(n - \sigma_1(n))), \\ y(n+1) = b(n)y(n) + \beta k(n)g(x(n - \tau_2(n)), \quad y(n - \sigma_2(n))). \end{cases} \tag{7}$$

In the real world, there are interactions between two species including prey and predator, competition, mutual interaction and so on. System (7) can describe many phenomena in the applied mathematical sciences [3,5,6,13,16]. In this paper, we consider the periodic phenomenon in this model and assume that $a(n)$, $b(n)$, $h(n)$, $k(n)$, $\tau_i(n)$, $\sigma_i(n)$ $(i = 1, 2)$ are T-periodic. Delay effects widely exist in the real world [17]. For system (7), $\tau_i(n)$ and $\sigma_i(n)$ $(i = 1, 2)$ represent delay.

We mainly consider the competitive system in this paper, and without loss of generality we assume that T is a positive integer with $T \geq 1$. $f(x(n), y(n))$, $g(x(n), y(n))$ are positive functions for all $n \in N[0, T-1]$, where N denotes natural numbers, and given $a < b$, $N[a, b] = \{a, a+1, \ldots, b\}$. The functions $f : R_+^2 \to R_+^2$ and $g : R_+^2 \to R_+^2$ are continuous. $a(n), b(n) : N[0, T-1] \to R_+$; $h(n), k(n) : N[0, T-1] \to R_-$, where R_- denotes the negative real numbers. $\tau_i(n), \sigma_i(n) : N[0, T-1] \to Z$ $(i = 1, 2)$, where Z denotes the integers. α and β will be assumed to be non-negative and treated as interaction parameters. To ensure the system to be coupled, α and β are considered in the set $\{(\alpha, \beta) | (\alpha, \beta) \in R_+^2 \backslash (0,0)\}$.

This paper is organized as follows. In Section 2, in order to get the existence of periodic positive solutions for a competitive system with two parameters, we give some preparations. In Section 3, we construct a special curve Δ, which divides the parameter space into two disjoint subsets Λ_1, Λ_2, then by using some lemmas and the degree theory, we prove that there are at least two T-periodic solutions in Λ_1, at least one T-periodic solution in Δ, and no periodic solution in Λ_2. In Section 4, we present conclusion and discussion.

2. Some preparation

In fact, system (7) has a T-periodic solution if and only if the following form

$$\begin{cases} x(n) = \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x(i-\tau_1(i)), \quad y(i-\sigma_1(i))), \\ y(n) = \beta \sum_{j=n}^{n+T-1} N(n,j)k(j)g(x(j-\tau_2(j)), \quad y(j-\sigma_2(j))) \end{cases} \tag{8}$$

has a T-periodic solution [14], where

$$M(n,i) = \frac{\prod_{s=i+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}, \quad i \in [n, n+T-1], \quad N(n,j) = \frac{\prod_{s=j+1}^{n+T-1} b(s)}{1 - \prod_{s=n}^{n+T-1} b(s)},$$
$$j \in [n, n+T-1].$$

In order to ensure the system has positive solutions, the following conditions are essential:

$$M(n,i)h(n) > 0 \quad \text{and} \quad N(n,j)k(n) > 0.$$

Then, we have the following four cases:

(i) When $M(n,i) > 0, N(n,j) > 0$, that is $0 < \prod_{s=n}^{n+T-1} a(s) < 1$, $0 < \prod_{s=n}^{n+T-1} b(s) < 1$, then $h(n) > 0$, $k(n) > 0$.
(ii) When $M(n,i) < 0, N(n,j) < 0$, that is $\prod_{s=n}^{n+T-1} a(s) > 1$, $\prod_{s=n}^{n+T-1} b(s) > 1$, then $h(n) < 0$, $k(n) < 0$.
(iii) When $M(n,i) > 0, N(n,j) < 0$, that is $0 < \prod_{s=n}^{n+T-1} a(s) < 1$, $\prod_{s=n}^{n+T-1} b(s) > 1$, then $h(n) > 0$, $k(n) < 0$.
(iv) When $M(n,i) < 0, N(n,j) > 0$, that is $\prod_{s=n}^{n+T-1} a(s) > 1$, $0 < \prod_{s=n}^{n+T-1} b(s) < 1$, then $h(n) < 0$, $k(n) > 0$.

Throughout this paper, we always assume that $\prod_{s=n}^{n+T-1} a(s) > 1$, $\prod_{s=n}^{n+T-1} b(s) > 1$, $h(n) < 0$, $k(n) < 0$, $n \in N[0, T-1]$, which means that the system we will study is a competitive system [2,8,9,15].

Definition 1. *A real function G is non-decreasing on R_+^2, if $G(x_1, y_1) \le G(x_2, y_2)$ for $(x_1, y_1) \le (x_2, y_2)$.*

From the biological point of view, we will need the following assumptions.

(H_1): $f(x, y)$ and $g(x, y)$ are non-decreasing and $f(0, 0) > 0$ and $g(0, 0) > 0$.
(H_2): $\lim_{x,y\to\infty}((f(x, y))/(x + y)) = \infty$ and $\lim_{x,y\to\infty}((g(x, y))/x + y) = \infty$

Definition 2. *A solution of system (7) corresponding to (a, b) implies that when $\alpha = a, \beta = b$, a vector function of the form $(x(n), y(n))$ satisfies system (7).*

Recall that system (7) has a T-periodic solution $(x(n), y(n))$ if and only if $(x(n), y(n))$ is a T-periodic solution of system (8). Therefore, we can transform our existence problem into a fixed point problem. Denote that

$$M_1 = \min_{i\in[n,n+T-1]} M(n, i)h(i) \le M(n, i)h(i) \le \max_{i\in[n,n+T-1]} M(n, i)h(i) = M_2,$$
$$N_1 = \min_{j\in[n,n+T-1]} N(n, j)k(j) \le N(n, j)k(j) \le \max_{j\in[n,n+T-1]} N(n, j)k(j) = N_2$$

and

$$\frac{M_1}{M_2} \le \frac{M(n, i)h(i)}{\max_{i\in[n,n+T-1]} M(n, i)h(i)} \le 1, \quad \frac{N_1}{N_2} \le \frac{N(n, j)k(j)}{\max_{j\in[n,n+T-1]} N(n, j)k(j)} \le 1.$$

Let X be the set of all real T-periodic sequences, which is endowed with the norm

$$\|u\| = \max_{n\in N[0,T-1]} |u(n)|.$$

Then X^2 is also a Banach space with the norm $\|(x, y)\| = \|x\| + \|y\|$.

Define two cones in X^2, respectively, by

$$\phi_1 = \{(x, y) \in X^2 : x(n), y(n) \ge 0, n \in N[0, T-1]\},$$
$$\phi_2 = \{(x, y) \in \phi_1 : x(n) + y(n) \ge \gamma\|(x, y)\|, n \in N[0, T-1]\},$$

where

$$\gamma = \min\left\{\frac{M_1}{M_2}, \frac{N_1}{N_2}\right\}.$$

Meanwhile, we define an operator $P : \phi_1 \to \phi_2$ and for each $(x, y) \in X^2$,

$$P_{\alpha,\beta}(x, y)(n) = (E_\alpha(x, y)(n), F_\beta(x, y)(n)),$$

where

$$E_\alpha(x,y)(n) = \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x(i-\tau_1(i)), y(i-\sigma_1(i))),$$

$$F_\beta(x,y)(n) = \beta \sum_{j=n}^{n+T-1} N(n,j)k(j)g(x(j-\tau_2(j)), y(j-\sigma_2(j))).$$

In the following, we will verify $P_{\alpha,\beta}\phi_1$ is contained in ϕ_2. It is easy to see that $P_{\alpha,\beta}$ is completely continuous and for $(x,y) \in \phi_1$,

$$E_\alpha(x,y)(n) = \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x(i-\tau_1(i)), y(i-\sigma_1(i)))$$

$$\leq \alpha M_2 \sum_{i=n}^{n+T-1} f(x(i-\tau_1(i)), y(i-\sigma_1(i))) \quad = \alpha M_2 \sum_{i=0}^{T-1} f(x(i-\tau_1(i)), y(i-\sigma_1(i)))$$

equals to

$$\frac{\|E_\alpha(x,y)\|}{M_2} \leq \alpha \sum_{i=0}^{T-1} f(x(i-\tau_1(i)), y(i-\sigma_1(i))).$$

And

$$E_\alpha(x,y)(n) = \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x(i-\tau_1(i)), y(i-\sigma_1(i)))$$

$$\geq \alpha M_1 \sum_{i=n}^{n+T-1} f(x(i-\tau_1(i)), y(i-\sigma_1(i))) \quad = \alpha M_1 \sum_{i=0}^{T-1} f(x(i-\tau_1(i)), y(i-\sigma_1(i)))$$

$$\geq \gamma \|E_\alpha(x,y)\|.$$

By the same method, we can obtain

$$F_\beta(x,y)(n) \geq \gamma \left\|F_\beta(x,y)\right\|.$$

Then

$$E_\alpha(x,y)(n) + F_\beta(x,y)(n) \geq \gamma\left(\|E_\alpha(x,y)\| + \left\|F_\beta(x,y)\right\|\right) \quad = \gamma\left\|(E_\alpha(x,y), F_\beta(x,y))\right\|,$$

which verifies that $P_{\alpha,\beta}\phi_1$ is contained in ϕ_2.

LEMMA 1. *Assume that* (H_2) *holds. For any compact subset* C *of* $R_+^2 \backslash \{(0,0)\}$, *any positive* T*-periodic solution of system (7) corresponding to* $(\alpha, \beta) \in C$ *is bounded, that is there exists a constant* b_C *such that* $\|(x,y)\| < b_C$.

Proof. Suppose on the contrary that there exists a sequence $\{(x_m, y_m)\}_{m\in N}$ of positive T-periodic solutions of system (7) about (α_m, β_m) such that $(\alpha_m, \beta_m) \in C$ for all m and

$\|(x_m, y_m)\| \to +\infty$ as $m \to +\infty$. Note that (x_m, y_m) satisfies system (8), so that $(x_m, y_m) \in \phi_2$. That is, $x_m + y_m \geq \gamma\|(x_m, y_m)\|$, where $m \geq 1$.

Now we assume that $\alpha_m > 0$ and $\beta_m \geq 0$ for sufficiently large m, and by (H_2), we may choose $Q_f > 0$, η and $m_0 > 1$, such that $f(x, y) \geq \eta(x + y)$ for all non-negative x, y which satisfy $x + y \geq Q_f$, $x_{m_0} + y_{m_0} \geq Q_f$ and $\gamma\eta M_1\alpha_{m_0} > 1$.

As a result, we can get

$$\begin{aligned}\|x_{m_0}\| &\geq x_{m_0}(n) = \alpha_{m_0}\sum_{i=n}^{n+T-1} M(n, i)h(i)f(x_{m_0}(i - \tau_1(i)), y_{m_0}(i - \sigma_1(i)))\\ &\geq \gamma\eta M_1\alpha_{m_0}\left(\|x_{m_0}\| + \|y_{m_0}\|\right) > \|x_{m_0}\|,\end{aligned}$$

which is impossible.

Similarly, when $\alpha_m \geq 0$ and $\beta_m > 0$ for sufficiently large m, by using $g_\infty = \infty$, the lemma can also be easily proved.

The proof is completed. □

Lemma 2. *Assume* (H_1) *holds, then system* (7) *has a positive T-periodic solution corresponding to some* (α_*, β_*) *satisfying* $\alpha_*, \beta_* > 0$.

Proof. Let

$$u(n) = \sum_{i=n}^{n+T-1} M(n, i)h(i), \quad v(n) = \sum_{j=n}^{n+T-1} N(n, j)k(j)$$

and

$$\begin{aligned}M_f &= \max_{n\in N[0,T-1]} f(x(i - \tau_1(i)), y(i - \sigma_1(i))) > 0,\\ M_g &= \max_{n\in N[0,T-1]} g(x(j - \tau_2(i)), y(j - \sigma_2(j))) > 0.\end{aligned}$$

Let $\alpha_*, \beta_* = ((1/M_f), (1/M_g)) > (0, 0)$, then we have

$$u(n) = \sum_{i=n}^{n+T-1} M(n, i)h(i) \quad \geq \alpha_* \sum_{i=n}^{n+T-1} M(n, i)h(i)f(u(i - \tau_1(i)), v(i - \sigma_1(i)))$$

and

$$v(n) = \sum_{j=n}^{n+T-1} N(n, j)k(j) \quad \geq \beta_* \sum_{j=n}^{n+T-1} N(n, j)h(j)f(u(j - \tau_2(j)), v(j - \sigma_2(j))).$$

Let

$$(\bar{x}_0(n), \bar{y}_0(n)) = (u(n), v(n)), \quad (\bar{x}_{m+1}(n), \bar{y}_{m+1}(n)) = P_{\alpha_*,\beta_*}(\bar{x}_m, \bar{y}_m)(n), \quad m = 0, 1, 2, \ldots$$

and

$$(\underline{x}_0(n), \underline{y}_0(n)) = (0,0), \quad (\underline{x}_{m+1}(n), \underline{y}_{m+1}(n)) = P_{\alpha_*,\beta_*}(\underline{x}_m, \underline{y}_m)(n), \quad m = 0,1,2, \ldots$$

Clearly, we have

$$(\bar{x}_0(n), \bar{y}_0(n)) \geq (\bar{x}_1(n), \bar{y}_1(n)) \geq \cdots \geq (\bar{x}_m(n), \bar{y}_m(n)(n)) \geq (\underline{x}_m(n), \underline{y}_m(n))$$

$$\geq \cdots \geq (\underline{x}_1(n), \underline{y}_1(n)) > (\underline{x}_0(n), \underline{y}_0(n)) = (0,0).$$

Let $(x(n), y(n)) = \lim_{m\to\infty} (\bar{x}_m(n), \bar{y}_m(n))$. Then we see that $(x(n), y(n))$ is a non-negative T-periodic solution of system (7), and we have $(x(n), y(n)) > (\underline{x}_1(n), \underline{y}_1(n)) > (\underline{x}_0(n), \underline{y}_0(n)) = (0,0)$.

The proof is completed. □

Lemma 3. *Assume (H_1) holds. If system (7) has a positive T-periodic solution corresponding to $(\bar{\alpha}, \bar{\beta}) > (0,0)$, then for any $(\alpha, \beta) \in R^2_+ \backslash \{(0,0)\}$ which satisfies $(\alpha, \beta) \leq (\bar{\alpha}, \bar{\beta})$, system (7) also has a positive T-periodic solution corresponding to (α, β).*

Proof. Let $(\bar{x}, \bar{y})$ be a positive T-periodic solution of system (7) corresponding to $(\bar{\alpha}, \bar{\beta})$. From (8) and (H_1), we have

$$\bar{x}(n) = \bar{\alpha} \sum_{i=n}^{n+T-1} M(n,i)h(i)f(\bar{x}(i-\tau_1(i)), \bar{y}(i-\sigma_1(i)))$$

$$\geq \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(\bar{x}(i-\tau_1(i)), \bar{y}(i-\sigma_1(i)))$$

and

$$\bar{y}(n) \geq \beta \sum_{j=n}^{n+T-1} N(n,j)k(j)g(\bar{x}(j-\tau_2(j)), \bar{y}(j-\sigma_2(j))).$$

Let $(\bar{x}_0(n), \bar{y}_0(n)) = (\bar{x}(n), \bar{y}(n))$, then

$$(\bar{x}_{m+1}(n), \bar{y}_{m+1}(n)) = P_{\alpha,\beta}(\bar{x}_m, \bar{y}_m)(n), \quad m = 0,1,2, \ldots$$

And $(\underline{x}_0(n), \underline{y}_0(n)) = (0,0)$, then

$$(\underline{x}_{m+1}(n), \underline{y}_{m+1}(n)) = P_{\alpha,\beta}(\underline{x}_m, \underline{y}_m)(n), \quad m = 0,1,2, \ldots$$

From the above analysis, we have

$$(\bar{x}_0(n), \bar{y}_0(n)) \geq (\bar{x}_1(n), \bar{y}_1(n)) \geq \cdots \geq (\bar{x}_m(n), \bar{y}_m(n)) \geq (\underline{x}_m(n), \underline{y}_m(n))$$

$$\geq \cdots \geq (\underline{x}_1(n), \underline{y}_1(n) > (\underline{x}_0(n), \underline{y}_0(n)) = (0,0).$$

Let $(x(n), y(n)) = \lim_{m\to\infty} (\bar{x}_m(n), \bar{y}_m(n))$, then $(x(n), y(n))$ is a non-negative T-periodic solution of (7), and thus we have $(x(n), y(n)) > (\underline{x}_1(n), \underline{y}_1(n)) > (\underline{x}_0(n), \underline{y}_0(n)) = (0,0)$.

The proof is completed. □

Let $\Pi = \{(\alpha, \beta) | (\alpha, \beta) \in R_+^2 \backslash (0,0)\}$ be the region such that system (7) has a positive T-periodic solution corresponding to (α, β). And by Lemma 2, Π contains the solution of system (7) corresponding to (α_*, β_*). So by Lemma 3, it contains the subset

$$\Pi_* = \{(\alpha, \beta) | (\alpha, \beta) > (0,0), \alpha \leq \alpha_*, \beta \leq \beta_*\}.$$

Then under conditions (H_1) and (H_2), we can show that Π is bounded.

Lemma 4. *Assume* (H_1) *and* (H_2) *hold, then* Π *is bounded above.*

Proof. Suppose on the contrary that there exists a sequence (x_m, y_m) of positive T-periodic solutions of system (7) corresponding to (α_n, β_n) such that $\lim_{n\to\infty} \alpha_n = \infty$ or $\lim_{n\to\infty} \beta_n = \infty$. If $\lim_{n\to\infty} \alpha_n = \infty$, then either there exists a subsequence (x_{m_j}, y_{m_j}) such that $(x_{m_j}, y_{m_j}) \to +\infty$ as $j \to \infty$ or there is a $\bar{G} > 0$ such that $\|(x_m, y_m)\| \leq \bar{G}$ for all m. Since $(x_m, y_m) \in \phi_2$, then we have

$$x_m(n) + y_m(n) \geq \gamma \|(x_m, y_m)\|.$$

By (H_2), there exists $Q_f > 0$ and some $\eta_1 > 0$ such that $f(x, y) \geq \eta_1(x + y)$ for all $x + y \geq Q_f$. From (H_1), we can get that there exists a $\eta_2 > 0$ such that $f(0,0) \geq \eta_2 G$. Let $\eta = \min\{\eta_1, \eta_2\}$. On the other hand, there exists a sequence $n_j \subset [0, T-1]$ such that $\|x_j\| = x(n_j)$. Thus, we can obtain that

$$\begin{aligned} a(n_j)\|x_j\| &= a(n_j)x(n_j) = x_j(n_j+1) - \alpha_j h(n_j) f(x(n_j - \tau_1(n_j)), y(n_j - \sigma_1(n_j))) \\ &\geq x_j(n_j+1) - \eta\alpha_j h(n_j)(x(n_j - \tau_1(n_j)) + y(n_j - \sigma_1(n_j))) \\ &\geq \|x_j\| - \eta\alpha_j h(n_j)\gamma n\|(x_j, y_j)\| \geq \|x_j\| - \eta\alpha_j h(n_j)\gamma\|x_j\| = \|x_j\|(1 - \eta\alpha_j h(n_j)\gamma). \end{aligned}$$

We have $\alpha_j \leq ((1 - a(n_j))/(\eta h(n_j)\gamma))$ which is in contradiction with $\lim_{n\to\infty} \alpha_n = \infty$. Similarly, when $\lim_{n\to\infty} \beta_n = \infty$, we can also get the result.

The proof is completed.

3. Main results

For the main results, we refer to the following classical works.

Lemma 5. [7]. *Let X be a Banach space with cone K. Let Ω be a bounded and open subset in X. Let $0 \in \Omega$ and $I : K \cap \bar{\Omega} \to K$ be condensing. Suppose that $Ix \neq \nu x$ for all $x \in K \cap \partial\Omega$ and all $\nu \geq 1$. Then $i(I, K \cap \Omega, K) = 1$.*

Lemma 6. [7]. *Let X be a Banach space and K a cone in X. For $r > 0$, define $K_r = \{x \in K : \|x < r\| < r\}$. Assume that $I : \overline{K_r} \to K$ is a compact map such that $Ix \neq x$ for $x \in \partial K_r$. If $\|x\| \leq \|Ix\|$ for $x \in \partial K_r$, then $i(I, K_r, K) = 0$.*

Then, for each $\theta \in [0, \pi/2]$, denote

$$L_\theta = \{(\alpha, \beta) | (\alpha, \beta) \in R_+^2 \backslash (0,0)\} = d(\sin\theta, \cos\theta), \quad d > 0.$$

The points near one end of this ray belong to Π_* and the points near the other end are outside Π, which implies that the set $\{d > 0 | d(\sin\theta, \cos\theta)\} \in \Pi$ is non-empty and bounded. Thus we can define

$$d_\theta^* = \sup\{d > 0 | d(\sin\theta, \cos\theta)\} \in \Pi$$

for each $\theta \in [0, \pi/2]$.

We can easily get $(\alpha_\theta^*, \beta_\theta^*) \in \Pi$, where $(\alpha_\theta^*, \beta_\theta^*) = d_\theta^*(\sin\theta, \cos\theta)$. Let $\{(\alpha_m, \beta_m)\}_{m=1}^\infty$ be an increasing sequence for $m \geq 1$ and converges to $(\alpha_\theta^*, \beta_\theta^*)$. For each m, let (x_m, y_m) be a positive T-periodic solution of system (7) corresponding to (α_m, β_m). By Lemma 1, we can get that the set (x_m, y_m) is uniformly bounded in X^2, which implies that the sequence (x_m, y_m) has a subsequence converging to $(x, y) \in X^2$. So we can obtain that (x, y) is a positive T-periodic solution of system (7) at $(\alpha_\theta^*, \beta_\theta^*)$.

Define ρ as $\rho(\theta) = \{(\alpha_\theta^*)^2 + (\beta_\theta^*)^2\}^{1/2} = d_\theta^*$. Without loss of generality, we assume that $\varphi \in (0, \pi/2)$ and B is an open neighbourhood containing $(\alpha_\varphi^*, \beta_\varphi^*)$ and contained in the interior of R_+^2. For any half-ray L_θ that passes through B, it is clear that there will be some points $(\tilde{\alpha}, \tilde{\beta})$ in L_θ such that $(\tilde{\alpha}, \tilde{\beta}) \leq (\alpha_\varphi^*, \beta_\varphi^*)$. From Lemma 3, there will be a positive T-periodic solution $(\tilde{x}, \tilde{y})$ of system (7) corresponding to $(\tilde{\alpha}, \tilde{\beta})$. If we pick the neighbourhood B such that in polar coordinates:

$$\{(r, \theta) | \rho(\varphi) - \varepsilon < r < \rho(\varphi) + \varepsilon, \varphi - \delta < \theta < \varphi + \delta\},$$

where ε and δ are sufficiently small positive numbers, then we can obtain that $\rho(\theta) > \rho(\varphi) - \varepsilon$. By symmetric arguments, we can also show that $\rho(\theta) < \rho(\varphi) + \varepsilon$. These arguments show that when θ and φ are sufficiently close, so is $\rho(\theta)$ and $\rho(\varphi)$.

Generalizing the above considerations as follows: under the conditions (H_1) and (H_2), there exists a continuous curve Δ (defined by ρ) joining some points $(\rho(0), 0)$ on the positive α-axis and some points $(0, \rho(\pi/2))$ on the positive β-axis and separating $R_+^2 \backslash (0,0)$ into two disjoint subsets Λ_1 and Λ_2 such that $(0, 0)$ is a boundary point of Λ_1 and system (7) has at least one positive T-periodic solution for $(\alpha, \beta) \in \Lambda_1 \cup \Delta$ and no positive T-periodic solution for $(\alpha, \beta) \in \Lambda_2$.

In the following, we will show that there are at least two periodic solutions for each $(\alpha, \beta) \in \Lambda_1$. Thus, we suppose that condition (H_1) holds and system (7) has a positive T-periodic solution $(\bar{x}, \bar{y})$ corresponding to $(\bar{\alpha}, \bar{\beta}) > (0, 0)$. Then by Lemma 3, system (7) also has a positive T-periodic solution $(x, y) < (\bar{x}, \bar{y})$ corresponding to $(\alpha, \beta) \in R_+^2 \backslash (0, 0)$ and $(\alpha, \beta) < (\bar{\alpha}, \bar{\beta})$.

Let(x^*, y^*) be a positive T-periodic solution of system (7) when $(\alpha^*, \beta^*) \in \Delta$. Then for $(\alpha, \beta) < (\alpha^*, \beta^*)$ and $(\alpha, \beta) \in R_+^2 \backslash (0, 0)$, there exists a $\varepsilon_0 > 0$ such that

$$f(x^*(i - \tau_1(i)) + \varepsilon, y^*(i - \sigma_1(i)) + \varepsilon) - f(x^*(i - \tau_1(i)), y^*(i - \sigma_1(i)))$$
$$< \frac{f(0,0)(\alpha^* - \alpha)}{\alpha}$$

and

$$g(x^*(j-\tau_2(j))+\varepsilon, y^*(j-\sigma_2(j))+\varepsilon) - g(x^*(j-\tau_2(j)), y^*(j-\sigma_2(j)))$$
$$< \frac{g(0,0)(\alpha^*-\alpha)}{\alpha}$$

for $n \in N[0, T-1]$ and $0 < \varepsilon < \varepsilon_0$.

As a result, we have

$$\alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i))+\varepsilon, y^*(i-\sigma_1(i))+\varepsilon)$$
$$-\alpha^* \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i)))$$
$$= \alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)[f(x^*(i-\tau_1(i))+\varepsilon, y^*(i-\sigma_1(i))+\varepsilon)$$
$$- f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i)))]$$
$$-(\alpha^*-\alpha) \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i)))$$
$$< f(0,0)(\alpha^*-\alpha) \sum_{i=n}^{n+T-1} M(n,i)h(i)$$
$$-(\alpha^*-\alpha) \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i)))$$
$$= (\alpha^*-\alpha) \sum_{i=n}^{n+T-1} M(n,i)h(i)[f(0,0) - f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i))] \le 0$$

and

$$\alpha \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i))+\varepsilon, y^*(i-\sigma_1(i))+\varepsilon)$$
$$\le \alpha^* \sum_{i=n}^{n+T-1} M(n,i)h(i)f(x^*(i-\tau_1(i)), y^*(i-\sigma_1(i))) \quad = x^*(n) < x^*(n)+\varepsilon.$$

Similarly, we have

$$\beta \sum_{j=n}^{n+T-1} N(n,j)h(j)f(x^*(j-\tau_2(j))+\varepsilon, y^*(j-\sigma_2(j))+\varepsilon)$$
$$\le \beta^* \sum_{j=n}^{n+T-1} N(n,j)h(j)f(x^*(j-\tau_2(j)), y^*(j-\sigma_2(j))) \quad = y^*(n) < y^*(n)+\varepsilon.$$

Let

$$x^*_\varepsilon(n) = x^*(n) + \varepsilon, \quad y^*_\varepsilon(n) = y^*(n) + \varepsilon$$

and

$$\Psi = \{(x, y) \in X^2 : -\varepsilon < x(n) < x^*(n) + \varepsilon, -\varepsilon < y(n) < y^*(n) + \varepsilon, n \in [0, T-1]\}.$$

We can see that Ψ is bounded and open in X, $0 \in \Psi$ and $I : \phi_2 \cap \bar{\Psi} \to \phi_2$ is condensing (since it is completely continuous). Let $(x, y) \in \phi_2 \cap \partial\Psi$. Then there exists n' such that $x(n') = x^*_\varepsilon(n')$ or $y(n') = y^*_\varepsilon(n')$. Suppose that $y(n') = y^*_\varepsilon(n')$, then by ($H_1$), we have that

$$F_\beta(x, y)(n') = \beta \sum_{j=n'}^{n'+T-1} N(n', j)k(j)g(x(j - \tau_2(j)), y(j - \sigma_2(j)))$$

$$\leq \beta \sum_{j=n'}^{n'+T-1} N(n', j)k(j)g(x^*_\varepsilon(j - \tau_2(j)), y^*_\varepsilon(j - \sigma_2(j))) \quad < y^*_\varepsilon(n') = y(n') \leq \nu y(n')$$

for all $\nu \geq 1$. Similarly, if $x(n') = x^*_\varepsilon(n')$, we can also get $E_\alpha(x, y)(n') < \nu x(n')$ for all $\nu \geq 1$. Thus $I_{\alpha,\beta}(x, y) \neq \nu(x, y)$ for $(x, y) \in \phi_2 \cap \partial\Psi$ and $\nu \geq 1$. In view of Lemma 5, we have $i(I, \Psi \cap \phi_2, \phi_2) = 1$. By ($H_2$), there exists $Q_f > 0$ such that $f(x, y) \geq \eta(x + y)$ for all $x + y \geq Q_f$, where η satisfies $\gamma\beta\eta N_1 > 1$.

Let $R = \max\{b_C, Q_f/\gamma, \|(x^*_\varepsilon, y^*_\varepsilon)\|\}$, where C is a closed rectangle in $R^2_+\backslash\{(0, 0)\}$ containing (α, β). Let $\phi_R = \{(x, y) \in \phi_2 : \|(x, y)\| < R\}$. Then by Lemma 6, $(x, y) \neq I_{\alpha,\beta}(x, y)$ for all $(x, y) \in \partial\phi_R$. Furthermore, if $(x, y) \in \partial\phi_R$, then $x(n) + y(n) \geq \gamma\|(x, y)\| \geq Q_f$. Thus, we have

$$F_\beta(x, y)(n) = \beta \sum_{j=n}^{n+T-1} N(n, j)k(j)g(x(j - \tau_2(j)), y(j - \sigma_2(j)))$$

$$\geq \gamma\beta\eta N_1\|(x, y)\| > \|(x, y)\|.$$

Therefore, from $\|I_{\alpha,\beta}(x, y)\| \geq \|F_\alpha(x, y)\| > \|(x, y)\|$ and Lemma 6, we obtain that $i(I_{\alpha,\beta}, \phi_R, \phi_2) = 0$. By the additivity of the topological degree, we have

$$0 = i(I_{\alpha,\beta}, \phi_R, \phi_2) = i(I_{\alpha,\beta}, \phi_2 \cap \Psi, \phi_2) + i(I_{\alpha,\beta}, \phi_R\backslash\overline{\phi_2 \cap \Psi}, \phi_2).$$

For $i(I_{\alpha,\beta}, \phi_2 \cap \Psi, \phi_2) = 1$, we get that $i(I_{\alpha,\beta}, \phi_R\backslash\overline{\phi_2 \cap \Psi}, \phi_2) = -1$. As a result, $I_{\alpha,\beta}$ has a fixed point on $\phi_2 \cap \Psi$ and other points on $\phi_R\backslash\overline{\phi_2 \cap \Psi}$.

Then, we can get the main theorem.

THEOREM 1. *Assume (H_1) and (H_2) hold, then there exist two disjoint subsets Λ_1 and Λ_2 which are obtained from separating $\{(\alpha, \beta)|(\alpha, \beta) \in R^2_+\backslash(0, 0)\}$ by a continuous curve Δ joining some points on the positive α-axis and some points on the positive β-axis, such that system (7) has at least two positive T-periodic solutions for $(\alpha, \beta) \in \Lambda_1$, at least one positive T-periodic solution for $(\alpha, \beta) \in \Delta$, and no positive T-periodic solution for $(\alpha, \beta) \in \Lambda_2$.*

4. Conclusion and discussion

In this paper, we investigated a nonlinear competitive system with two parameters. By constructing a curve Δ, we divide the parameter space $\Pi = \{(\alpha, \beta) | (\alpha, \beta) \in R_+^2 \backslash (0,0)\}$ into two disjoint subsets Λ_1 and Λ_2. Moreover, by using the method of upper and lower solutions and the degree theory, we obtained that there are at least two positive periodic solutions for $(\alpha, \beta) \in \Lambda_1$, at least one positive periodic solution for $(\alpha, \beta) \in \Delta$ and no positive periodic solution for $(\alpha, \beta) \in \Lambda_2$.

In Ref. [18], Wu and Liu considered neutral difference systems depending on two parameters and studied the existence, multiplicity and non-existence of periodic solutions. Compared with their work, our paper has three main differences. First, we construct a special curve Δ, which joins some point $(\rho(0), 0)$ on the positive α-axis and some point $(0, \rho(\pi/2))$ on the positive β-axis, and which exactly divides the parameter space into two disjoint domains Λ_1 and Λ_2. Domain Λ_1 consists of boundary, but does not consist of $(0,0)$. However, in their paper, the continuous curve Γ is defined as $\Gamma = \{\lambda(\theta), \mu(\theta) : \theta \in (0, \pi/2)\}$, the domain they investigated does not include the boundary. Second, our results are still valid when $\alpha = 0, \beta \neq 0$ or $\beta = 0, \alpha \neq 0$. However, in their paper, the results do not hold for the two particular cases. Finally, we obtain the existence of multiple solutions only by the assumptions (H_1) and (H_2), while they got the existence of multiple solutions by adding another assumption.

From the discussion above, it can be concluded that there are multiple positive periodic solutions in the competitive system which is consistent with the findings in the real world. Moreover, the obtained results can also be extended to mutual systems and predator–prey systems.

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