Information Sciences 328 (2016) 359–377

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

A model for type-2 fuzzy rough sets

Juan Lu [a,b], De-Yu Li [a,*], Yan-Hui Zhai [a], Hua Li [a], He-Xiang Bai [a]

[a] School of Computer and Information Technology, Shanxi University, Taiyuan, Shanxi 030006, PR China
[b] School of Science, North University of China, Taiyuan, Shanxi 030051, PR China

A R T I C L E   I N F O

Article history:
Received 5 March 2014
Revised 27 June 2015
Accepted 23 August 2015
Available online 5 September 2015

Keywords:
Type-2 fuzzy rough set
Representation Theorem
Granular type-2 fuzzy set
Lower approximation operator
Upper approximation operator

A B S T R A C T

Rough set theory is an important approach to granular computing. Type-1 fuzzy set theory permits the gradual assessment of the memberships of elements in a set. Hybridization of these assessments results in a fuzzy rough set theory. Type-2 fuzzy sets possess many advantages over type-1 fuzzy sets because their membership functions are themselves fuzzy, which makes it possible to model and minimize the effects of uncertainty in type-1 fuzzy logic systems. Existing definitions of type-2 fuzzy rough sets are based on vertical-slice or α-plane representations of type-2 fuzzy sets, and the granular structure of type-2 fuzzy rough sets has not been discussed. In this paper, a definition of type-2 fuzzy rough sets based on a wavy-slice representation of type-2 fuzzy sets is given. Then the concepts of granular type-2 fuzzy sets are proposed, and their properties are investigated. Finally, granular type-2 fuzzy sets are used to describe the granular structures of the lower and upper approximations of a type-2 fuzzy set, and an example of attribute reduction is given.

1. Introduction

Since the theory of type-1 fuzzy sets was proposed by Zadeh in 1965 [21], it has been widely used in many areas of artificial intelligence. However, according to Mendel [10], there exist at least four sources of uncertainty in type-1 fuzzy logic systems: (1) the meanings of the words used in rule antecedents and consequents may be uncertain (the same word may mean different things to different people); (2) the consequents may have a histogram of values associated with them, especially when knowledge is extracted from a group of experts, all of whom do not agree with each other; (3) the measurements that activate a type-1 fuzzy logic system may be noisy and therefore uncertain; and (4) the data used to tune the type-1 fuzzy logic system parameters may also be noisy. All these uncertainties lead to uncertain fuzzy set membership functions.

Type-2 fuzzy sets were introduced by Zadeh [22] as an extension of the concept of type-1 fuzzy sets (ordinary fuzzy sets). Type-2 fuzzy sets can be used to describe the four kinds of uncertainty listed above because the membership functions of type-2 fuzzy sets are themselves fuzzy. However, type-2 fuzzy sets are nowhere near as widely used as type-1 fuzzy sets.

The secondary membership functions for general type-2 fuzzy sets are too difficult to construct and too complex to compute, and only a special kind of type-2 fuzzy set, the interval type-2 fuzzy set, is widely used because its secondary membership grades are equal to one. In other words, the membership degree of every element in the universe is characterized by a sub-interval of [0, 1], and any of the values in the sub-interval can be assigned as the membership degree, with each value having the same

* Corresponding author. Tel.: +86 351 7018775.
E-mail addresses: nancylu0312@163.com (J. Lu), lidy@sxu.edu.cn, lidysxu@163.com (D.-Y. Li), chai_yanhui@163.com (Y.-H. Zhai), hualisjzri@sina.com (H. Li), baihx@sxu.edu.cn (H.-X. Bai).

http://dx.doi.org/10.1016/j.ins.2015.08.045

probability. However, in practice, the probabilities of the values in the sub-interval may follow another distribution, such as the normal or triangular distribution [9].

In 1982, Pawlak proposed the theory of rough sets as a new mathematical tool for reasoning about data. Rough set theory has been under development for thirty years and has been successfully used in various fields of artificial intelligence such as expert systems, machine learning, pattern recognition, decision analysis, process control and knowledge discovery in databases [15]. Traditional rough set theory is based on an equivalence relation, which seems to be a very restrictive condition that may limit the applications of the rough set model. For example, the values of attributes may be both symbolic and real-valued, in which case they cannot be manipulated by traditional rough set theory. Two close values may differ only as a result of noise, but in traditional rough set theory, they may be considered to be of different orders of magnitude. To overcome these shortcomings, Dubois and Prade [4] combined fuzzy sets and rough sets by proposing definitions of rough fuzzy sets and fuzzy rough sets in 1990, after which many studies were carried out in the field of fuzzy rough sets. Shen and Jensen [18] proposed an approach that integrates a fuzzy rule induction algorithm with a fuzzy rough method for feature selection. Jensen and Shen [5] provided an interval-valued approach for fuzzy rough feature selection, which could handle missing values and uncertainties that could not be modeled by a type-1 approach. Wu et al. [20] proposed an attribute reduction method within the interval type-2 fuzzy rough set framework and presented the properties of interval type-2 fuzzy rough sets. To date, most research in fuzzy rough sets has been restricted to ordinary (type-1) fuzzy environments and interval type-2 fuzzy environments [1,3,6,7,17,23–25,27,28].

The point-valued representation is usually the starting point for understanding or describing a general type-2 fuzzy set, but it does not seem to be useful for much of anything else [13]. In addition, there are three other popular representations for a type-2 fuzzy set: the vertical-slice representation, the wavy-slice representation (which is also called the Mendel-John representation or an embedded type-2 fuzzy set representation), and the α-plane representation. Based on vertical-slice representations of type-2 fuzzy sets, Wang [19] investigated type-2 fuzzy rough sets on two finite universes of discourse using both constructive and axiomatic approaches and discussed the topological properties of type-2 fuzzy rough sets. Using α-plane representation theory, Zhao and Xiao [26] presented definitions of general type-2 fuzzy rough sets and studied some basic properties of upper and lower approximation operators. In addition, they examined the connections between special general type-2 fuzzy relations and general type-2 fuzzy rough upper and lower approximation operators and characterized various classes of general type-2 fuzzy rough approximation operators using an axiomatic approach. Many properties were proposed in [19] and [26], but no discussion of the granular structure of type-2 fuzzy rough sets was included.

According to Pedrycz, “Information granules are intuitively appealing constructs, which play a pivotal role in human cognitive and decision-making activities. We perceive complex phenomena by organizing existing knowledge along with available experimental evidence and structuring them in a form of some meaningful, semantically sound entities, which are central to all ensuing processes of describing the world, reasoning about the environment, and supporting decision-making activities” [16]. In classical rough set theory, lower and upper approximations are defined as unions of certain sets, exhibiting a clear granular structure over sets. Chen et al. [2] proposed the concept of granular fuzzy sets based on fuzzy similarity relations and described the granular structures of the lower and upper approximations of a fuzzy set within the framework of granular computing. The wavy-slice representation of a type-2 fuzzy set in terms of embedded type-2 fuzzy sets is most valuable in theoretical studies because it quickly leads to the structure of the solution to a new problem. To discuss the granular structure of type-2 fuzzy rough sets, a model for type-2 fuzzy rough sets is proposed here using the wavy-slice representation of a type-2 fuzzy set presented by Mendel and John [12]. Here the conclusions of Chen et al. are extended by proposing the concept of granular type-2 fuzzy sets and investigating their properties. Then these granular type-2 fuzzy sets are used to describe the granular structures of the lower and upper approximations of a type-2 fuzzy set. Finally, an example of attribute reduction within a type-2 fuzzy rough framework is presented.

The rest of this paper is organized as follows. Fundamental concepts and properties that will be used in this paper are reviewed in Section 2. Section 3 introduces the definition of a type-2 fuzzy rough set based on the wavy-slice representation. In Section 4, the granular structure of type-2 fuzzy rough sets is discussed using granular type-2 fuzzy sets. Conclusions are presented in Section 5.

2. Preliminaries

This section will review some basic notions and properties related to type-2 fuzzy sets, rough sets, and fuzzy rough sets.

2.1. Type-2 fuzzy sets

Definition 1 [12]**.** Let X be a nonempty universe of discourse. A type-2 fuzzy set, $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | x \in X, u \in J_x \subseteq [0, 1]\},$$

in which $0 \le \mu_{\tilde{A}}(x, u) \le 1$. $\tilde{A}$ can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u)/(x, u), J_x \subseteq [0, 1], \quad (1)$$

where $\int\int$ denotes union over all admissible x and u. The class of all type-2 fuzzy sets of the universe X is denoted by $\tilde{F}(X)$.

Definition 2 [12]**.** At each value of x, say $x = x'$, $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\mu_{\tilde{A}}(x, u)$, i.e.,

$$\mu_{\tilde{A}}(x = x', u) \equiv \mu_{\tilde{A}}(x') \equiv \int_{u \in J_{x'}} f_{x'}(u)/u, J_{x'} \subseteq [0, 1],$$

where $f_{x'} = \mu_{\tilde{A}}(x')$ and $0 \le f_{x'}(u) \le 1$. It is permissible to drop the prime notation on $\mu_{\tilde{A}}(x')$, and to refer to $\mu_{\tilde{A}}(x)$ as a secondary membership function; it is a type-1 fuzzy set, also referred to as a secondary set. The amplitude of a secondary membership function is called a secondary grade.

A type-2 fuzzy set can be reinterpreted as the union of all secondary sets, i.e.,

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\},$$

or, as

$$\tilde{A} = \int_{x \in X} \mu_{\tilde{A}}(x)/x = \int_{x \in X} \left[\int_{u \in J_x} f_x(u)/u \right] /x.$$

Definition 3 [12]**.** The domain of a secondary membership function is called the primary membership of x. In Definition 2, $J_{x'}$ is the primary membership of x'. Uncertainty in the primary membership of a type-2 fuzzy set $\tilde{A}$ consists of a bounded region called the footprint of uncertainty (FOU), which is the union of all primary memberships, i.e.,

$$\text{FOU}(\tilde{A}) = \bigcup_{x \in X} J_x.$$

In 2002, Mendel and John [12] presented a new representation for type-2 fuzzy sets to make these sets much easier to understand and to work with. They assumed that X and J_x are both discrete (either by problem formulation, or by discretization of continuous universes of discourse), that X has been discretized into N values, $x_1, \ldots, x_N$, and that at each of these values, J_{x_i} has been discretized into M_i values. For simplicity, the definition of a discrete type-2 fuzzy set is first introduced.

Definition 4 [11]**.** A partially discrete type-2 fuzzy set is one whose primary variable is discrete (sampled), but whose secondary membership functions are continuous, whereas a discrete type-2 fuzzy set is one whose primary variable and secondary membership functions are discrete (sampled).

A discrete type-2 fuzzy set $\tilde{A}$ can be expressed as:

$$\tilde{A} = \sum_{x \in X} \mu_{\tilde{A}}(x)/x = \sum_{x \in X} \left[\sum_{u \in J_x} f_x(u)/u \right] /x = \sum_{i=1}^{N} \left[\sum_{u \in J_{x_i}} f_{x_i}(u)/u \right] /x_i$$
$$= \left[\sum_{k=1}^{M_1} f_{x_1}(u_{1k})/u_{1k} \right] /x_1 + \cdots + \left[\sum_{k=1}^{M_N} f_{x_N}(u_{Nk})/u_{Nk} \right] /x_N.$$

In this equation, $+$ also denotes union. Note that x has been discretized into N values and that at each value x_i, u has been discretized into M_i values.

Definition 5 [12]**.** For a discrete type-2 fuzzy set $\tilde{A}$, an embedded type-2 set $\tilde{A}_e$ has N elements, where $\tilde{A}_e$ contains exactly one element from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, each with its associated secondary grade, namely $f_{x_1}(u_1), f_{x_2}(u_2), \ldots, f_{x_N}(u_N)$, i.e.,

$$\tilde{A}_e = \sum_{i=1}^{N} [f_{x_i}(u_i)/u_i]/x_i, \quad u_i \in J_{x_i} \subseteq [0, 1].$$

Set $\tilde{A}_e$ is embedded in $\tilde{A}$, and there are a total of $\prod_{i=1}^{N} M_i \, \tilde{A}_e$.

Definition 6 [12]**.** For a discrete type-2 fuzzy set $\tilde{A}$, an embedded type-1 set A_e has N elements, one each from $J_{x_1}, J_{x_2}, \ldots, J_{x_N}$, namely $u_1, u_2, \ldots, u_N$, i.e.,

$$A_e = u_i/x_i, \quad u_i \in J_{x_i} \subseteq [0, 1].$$

Set A_e is the union of all the primary memberships of set $\tilde{A}_e$, and there are a total of $\prod_{i=1}^{N} M_i \, A_e$.

Theorem 1 (Representation Theorem [12])**.** *Let $\tilde{A}_e^j$ denote the jth embedded type-2 set for a discrete type-2 fuzzy set $\tilde{A}$, i.e.,*

$$\tilde{A}_e^j \equiv \left\{ \left(u_i^j, f_{x_i}\left(u_i^j\right)\right), i = 1, \ldots, N \right\}$$

where $u_i^j \in \{u_{ik}, k = 1, \ldots, M_i\}$. Then $\tilde{A}$ can be represented as the union of its embedded type-2 sets, i.e.,

$$\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j$$

where $n = \prod_{i=1}^{N} M_i$.

The Representation Theorem provides a new representation for a type-2 fuzzy set in terms of so-called wavy slices.

Theorem 2 [12]. *Consider two discrete type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$, where*

$$\tilde{A} = \sum_{x\in X} \mu_{\tilde{A}}(x)/x = \sum_{x\in X}\left[\sum_{u\in J_x^u} f_x(u)/u\right]/x, \quad J_x^u \subseteq [0,1]$$

and

$$\tilde{B} = \sum_{x\in X} \mu_{\tilde{B}}(x)/x = \sum_{x\in X}\left[\sum_{w\in J_x^w} g_x(w)/w\right]/x, \quad J_x^w \subseteq [0,1].$$

Using the Representation Theorem, these can be expressed as:

$$\tilde{A} = \sum_{j=1}^{n_A} \tilde{A}_e^j = \sum_{j=1}^{n_A}\sum_{i=1}^{N} \frac{f_{x_i}(u_i^j)/u_i^j}{x_i}$$

and

$$\tilde{B} = \sum_{j=1}^{n_B} \tilde{B}_e^j = \sum_{j=1}^{n_B}\sum_{i=1}^{N} \frac{g_{x_i}(w_i^j)/w_i^j}{x_i}.$$

The union of two type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ is given as:

$$\tilde{A}\bigcup\tilde{B} = \sum_{j=1}^{n_A}\sum_{i=1}^{n_B}\left\{[f_{x_1}(u_1^j)\star g_{x_1}(w_1^i)/u_1^j \vee w_1^i]/x_1 + \cdots + [f_{x_N}(u_N^j)\star g_{x_N}(w_N^i)/u_N^j \vee w_N^i]/x_N\right\},$$

and the intersection of two type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$ is given as:

$$\tilde{A}\bigcap\tilde{B} = \sum_{j=1}^{n_A}\sum_{i=1}^{n_B}\left\{[f_{x_1}(u_1^j)\star g_{x_1}(w_1^i)/u_1^j \wedge w_1^i]/x_1 + \cdots + [f_{x_N}(u_N^j)\star g_{x_N}(w_N^i)/u_N^j \wedge w_N^i]/x_N\right\},$$

where $\star$ is a t-norm. In the following, $\star$ is taken as the largest t-norm, i.e., the minimum. The complement of the type-2 fuzzy set $\tilde{A}$ is given as:

$$(\tilde{A})^c = \sum_{j=1}^{n_A}\left(\sum_{i=1}^{N}[f_{x_i}(u_i^j)/(1-u_i^j)]/x_i\right)$$

or

$$(\tilde{A})^c = \sum_{i=1}^{N}\left(\sum_{j=1}^{M_i}[f_{x_i}(u_i^j)/(1-u_i^j)]\right)/x_i = \sum_{i=1}^{N}\neg\tilde{A}(x_i)/x_i,$$

where $\neg$ denotes the so-called negation operation and

$$\neg\tilde{A}(x_i) = \sum_{j=1}^{M_i}[f_{x_i}(u_i^j)/(1-u_i^j)]$$

is the negation of the secondary membership function $\tilde{A}(x_i)$.

Because the union and intersection of $\tilde{A}$ and $\tilde{B}$ are still type-2 fuzzy sets, an expression for $\mu_{\tilde{A}\cup\tilde{B}}(x)$ and $\mu_{\tilde{A}\cap\tilde{B}}(x)$ can be obtained as follows:

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \sum_{u\in J_x^u}\sum_{w\in J_x^w} f_x(u)\star g_x(w)/u\vee w \equiv \mu_{\tilde{A}}(x)\sqcup\mu_{\tilde{B}}(x),$$
$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \sum_{u\in J_x^u}\sum_{w\in J_x^w} f_x(u)\star g_x(w)/u\wedge w \equiv \mu_{\tilde{A}}(x)\sqcap\mu_{\tilde{B}}(x),$$

where $\sqcup$ denotes the so-called *join* operation and $\sqcap$ denotes the so-called *meet* operation. The notations $\mu_{\tilde{A}}(x)\sqcup\mu_{\tilde{B}}(x)$ and $\mu_{\tilde{A}}(x)\sqcap\mu_{\tilde{B}}(x)$ are used here to indicate the join and meet operations between the secondary membership functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$.

Definition 7. For two type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$, $\tilde{A}\subseteq\tilde{B}$ if and only if $\tilde{A}\cap\tilde{B}=\tilde{A}$.

For two discrete type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$, if they have unique embedded type-2 sets, i.e.,

$$\tilde{A} = \sum_{i=1}^{N} \frac{f_{x_i}(u_i)/u_i}{x_i} \text{ and } \tilde{B} = \sum_{i=1}^{N} \frac{g_{x_i}(w_i)/w_i}{x_i},$$

then $\tilde{A} \subseteq \tilde{B}$ if and only if $u_i \le w_i$ and $f_{x_i}(u_i) \le g_{x_i}(w_i)$ $(\forall i = 1, \ldots, N)$.

For a family of discrete type-2 fuzzy sets with unique embedded type-2 sets

$$\tilde{A}^{(m)} = \sum_{i=1}^{N} \frac{f_{x_i}^{(m)}(u_i^{(m)})/u_i^{(m)}}{x_i}, \quad m \in \theta,$$

where θ is a finite index set, the union of these type-2 fuzzy sets is:

$$\bigcup_{m\in\theta} \tilde{A}^{(m)} = \sum_{i=1}^{N} \frac{\bigwedge_{m\in\theta} f_{x_i}^{(m)}(u_i^{(m)})/\bigvee_{m\in\theta} u_i^{(m)}}{x_i},$$

and the intersection of these type-2 fuzzy sets is:

$$\bigcap_{m\in\theta} \tilde{A}^{(m)} = \sum_{i=1}^{N} \frac{\bigwedge_{m\in\theta} f_{x_i}^{(m)}(u_i^{(m)})/\bigwedge_{m\in\theta} u_i^{(m)}}{x_i}.$$

Example 1. Consider three type-2 fuzzy sets on $X = \{x_1, x_2, x_3, x_4\}$:

$$\tilde{A}^{(1)} = \frac{1/0}{x_1} + \frac{0.8/0.3}{x_2} + \frac{1/0.6}{x_3} + \frac{0.5/0.2}{x_4},$$
$$\tilde{A}^{(2)} = \frac{1/0.4}{x_1} + \frac{0.7/0.4}{x_2} + \frac{1/1}{x_3} + \frac{1/0.5}{x_4},$$
$$\tilde{A}^{(3)} = \frac{0.5/1}{x_1} + \frac{1/0.8}{x_2} + \frac{0.9/0.3}{x_3} + \frac{0.5/0}{x_4},$$

then

$$\tilde{A}^{(1)} \cup \tilde{A}^{(2)} \cup \tilde{A}^{(3)} = \frac{0.5/1}{x_1} + \frac{0.7/0.8}{x_2} + \frac{0.9/1}{x_3} + \frac{0.5/0.5}{x_4},$$
$$\tilde{A}^{(1)} \cap \tilde{A}^{(2)} \cap \tilde{A}^{(3)} = \frac{0.5/0}{x_1} + \frac{0.7/0.3}{x_2} + \frac{0.9/0.3}{x_3} + \frac{0.5/0}{x_4}.$$

Definition 8 [8]**.** Let X and Y be two nonempty universes. A type-2 fuzzy set $\tilde{R} \in \tilde{F}(X \times Y)$ is defined as a type-2 fuzzy relation from X to Y. If $X = Y$, then $\tilde{R}$ is called a type-2 fuzzy relation on X.

Definition 9. Let X be a nonempty universe, and let $\tilde{R}$ be a type-2 fuzzy relation on X.

1. $\tilde{R}$ is reflexive if for any $x \in X$, $\tilde{R}(x, x) = 1/1$.
2. $\tilde{R}$ is symmetric if for any $x, y \in X$, $\tilde{R}(x, y) = \tilde{R}(y, x)$.
3. $\tilde{R}$ is transitive if for any $x, y \in X$, $\tilde{R}(x, y) \ge \sqcup_{z\in X}(\tilde{R}(x, z) \sqcap \tilde{R}(z, y))$.

A type-2 fuzzy relation $\tilde{R}$ on X is called a type-2 fuzzy similarity relation if $\tilde{R}$ is reflexive, symmetric, and transitive.

Example 2. Let $X = \{a, b, c\}$, then

$$\tilde{R} = \frac{1/1}{(a,a)} + \frac{1/0.8}{(a,b)} + \frac{1/0.8}{(a,c)} + \frac{1/0.8}{(b,a)} + \frac{1/1}{(b,b)} + \frac{0.5/0.85}{(b,c)} + \frac{1/0.8}{(c,a)} + \frac{0.5/0.85}{(c,b)} + \frac{1/1}{(c,c)}$$

is a type-2 fuzzy similarity relation.

Assume a discrete type-2 fuzzy relation $\tilde{R}$ from X to Y, where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The lth embedded type-2 relation $\tilde{R}_e^l$ can be expressed as:

$$\tilde{R}_e^l = \{(u_{ij}^l, f_{x_i y_j}(u_{ij}^l)), i = 1, \ldots, n; j = 1, \ldots, m\}$$

where $u_{ij}^l \in \{u_{ijk} | k = 1, \ldots, M_{ij}\}$. The u_{ijk} represent the members of the primary membership set for a given pair (x_i, y_j), and M_{ij} are the number of u_{ijk} in that vertical slice.

The Representation Theorem shows that a discrete type-2 fuzzy set is the union of its embedded type-2 sets, which is also true for a discrete type-2 fuzzy relation:

$$\tilde{R} = \sum_{l=1}^{n_R} \tilde{R}_e^l \quad \text{where} \quad n_R \equiv \prod_{i=1}^{n} \prod_{j=1}^{m} M_{ij}$$

because a type-2 fuzzy relation is a type-2 fuzzy set in essence.

In 1975, Zadeh proposed the Extension Principle for fuzzy sets [22], which is essentially a basic identity enabling the domain of a mapping or a relation to be extended from points in U to fuzzy subsets in U. This paper reviews the statement of the Extension Principle and presents a version following Mendel and John [12].

Let $A_1, A_2, \ldots, A_r$ be type-1 fuzzy sets in $X_1, X_2, \ldots, X_r$, respectively. Then a type-1 fuzzy set B can be induced on Y from the r type-1 fuzzy sets $A_1, A_2, \ldots, A_r$ through f, i.e., $B = f(A_1, A_2, \ldots, A_r)$, such that:

$$\mu_B(y) = \begin{cases} \sup_{(x_1,\ldots,x_r)\in f^{-1}(y)} \min\{\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

where $f^{-1}(y)$ denotes the set of all points $x_1 \in X_1, \ldots, x_r \in X_r$ such that $f(x_1, \ldots, x_r) = y$.

2.2. Rough sets

Let X be a finite and nonempty universe, and let $R \subseteq X \times X$ be an equivalence relation on X, i.e., R is reflexive, symmetric, and transitive. The pair (X, R) is called an approximation space. A mapping $[\cdot]_R: X \to 2^X$ can be defined as: $[x]_R = \{y \in X : (x, y) \in R\}$. The subset $[x]_R$ is the equivalence class containing x. The family of all equivalence classes $\{[x]_R: x \in X\}$ is known as the quotient set and is denoted by X/R. The quotient set X/R defines a partition of the universe X, namely a family of pairwise disjoint subsets whose union is the universe. If two elements x and y in X belong to the same equivalence class, x and y are said to be indiscernible. Under the equivalence relation, a coarsened view of the universe is obtained. Given an arbitrary set $A \subseteq X$, it may not be possible to describe A precisely in (X, R), in which case A may be characterized by a pair of lower and upper approximations defined as follows:

$$\underline{R}A = \cup\{[x]_R : [x]_R \subseteq A\} = \{x \in X : [x]_R \subseteq A\},$$
$$\overline{R}A = \cup\{[x]_R : [x]_R \cap A \neq \emptyset\} = \{x \in X : [x]_R \cap A \neq \emptyset\} = \cup\{[x]_R : x \in A\}.$$

The lower and upper approximations have another equivalent description obtained by using the intersections of complements of certain equivalence classes [2]. Suppose that R is an equivalence relation on X, then for $A \subseteq X$:

$$\underline{R}A = \cap\{([x]_R)^c : x \in A^c\}, \quad \overline{R}A = \cap\{([x]_R)^c : A \subseteq ([x]_R)^c\}.$$

2.3. Fuzzy rough sets

It is widely accepted that the concepts of fuzzy sets and rough sets are related but distinct and that both arise because of uncertainty in knowledge or data. Fuzzy rough sets encapsulate these two concepts and are able to deal better with uncertainty. Dubois and Prade [4] pioneered the concept of rough fuzzy sets and fuzzy rough sets. Here, the definition of fuzzy rough sets proposed by Dubois and Prade is briefly introduced with some notations modified.

Definition 10. Let X be a nonempty universe, R be a fuzzy binary relation on X, and $\mathcal{F}(X)$ be the fuzzy power set of X. For any $A \in \mathcal{F}(X)$, the fuzzy rough set of A is a pair $(R_*(A), R^*(A))$ such that for every $x \in X$:

$$R^*(A)(x) \equiv \sup_{y\in X} \min\{R(x, y), \mu_A(y)\}, \tag{2}$$

$$R_*(A)(x) \equiv \inf_{y\in X} \max\{1 - R(x, y), \mu_A(y)\}. \tag{3}$$

3. Type-2 fuzzy rough sets

Rough set theory is a mathematical approach to imperfect knowledge, whose methodology is concerned with the classification and analysis of imprecise, uncertain, or incomplete information and knowledge. In this section, type-2 fuzzy rough sets will be defined by combining rough sets and type-2 fuzzy sets.

Let $\tilde{A}$ be a discrete type-2 fuzzy set on X and $\tilde{R}$ be a discrete type-2 fuzzy relation on X, where $X = \{x_1, x_2, \ldots, x_n\}$. By the Representation Theorem, $\tilde{A}$ and $\tilde{R}$ can be expressed as: $\tilde{A} = \sum_{j=1}^{n_A} \tilde{A}_e^j$ and $\tilde{R} = \sum_{l=1}^{n_R} \tilde{R}_e^l$, where

$$\tilde{A}_e^j = \sum_{i=1}^{N} [f_{x_i}(u_i^j)/u_i^j]/x_i, \quad x_i \in X, u_i^j \in J_{x_i} \subseteq [0, 1] \tag{4}$$

and

$$\tilde{R}_e^l = \sum_{p,q=1}^{N} [g_{(x_p,y_q)}(v_{pq}^l)/v_{pq}^l]/(x_p, y_q), \quad (x_p, y_q) \in X \times X, v_{pq}^l \in J_{(x_p,y_q)} \subseteq [0, 1] \tag{5}$$

are embedded type-2 sets.

The upper and lower approximation operators can be defined as follows:

$$\overline{\tilde{R}}(\tilde{A}) = \sum_{l=1}^{n_R}\sum_{j=1}^{n_A} \overline{\tilde{R}^l_e}(\tilde{A}^j_e), \tag{6}$$

$$\underline{\tilde{R}}(\tilde{A}) = \sum_{l=1}^{n_R}\sum_{j=1}^{n_A} \underline{\tilde{R}^l_e}(\tilde{A}^j_e). \tag{7}$$

It is clear that to define the upper and lower approximation operators, it is necessary to evaluate $\overline{\tilde{R}^l_e}(\tilde{A}^j_e)$ and $\underline{\tilde{R}^l_e}(\tilde{A}^j_e)(\forall\, l, j)$.

Because a type-1 fuzzy set can be regarded as a special type-2 fuzzy set with all secondary grades equal to unity, it seems sensible to use the definition of a (type-1) fuzzy rough set as a starting point for the discussion of type-2 fuzzy rough sets.

Recall that A^j_e and R^l_e are the embedded type-1 sets associated with $\tilde{A}^j_e$ and $\tilde{R}^l_e$, i.e.,

$$A^j_e = \sum_{i=1}^{N} u^j_i/x_i, \quad x_i \in X, u^j_i \in J_{x_i} \subseteq [0, 1],$$

$$R^l_e = \sum_{p,q=1}^{N} \upsilon^l_{pq}/(x_p, y_q), \quad (x_p, y_q) \in X \times X, \upsilon^l_{pq} \in J_{(x_p,y_q)} \subseteq [0, 1].$$

By (2) and (3),

$$\begin{aligned}\overline{R^l_e}(A^j_e) &= \sum_{i=1}^{N}\left\{\bigvee_{m=1}^{N}[R^l_e(x_i, y_m) \wedge A^j_e(y_m)]\right\}/x_i \\ &= \sum_{i=1}^{N}\left[\bigvee_{m=1}^{N}(\upsilon^l_{im} \wedge u^j_m)\right]/x_i, \\ \underline{R^l_e}(A^j_e) &= \sum_{i=1}^{N}\left\{\bigwedge_{m=1}^{N}[(1 - R^l_e(x_i, y_m)) \vee A^j_e(y_m)]\right\}/x_i \\ &= \sum_{i=1}^{N}\left\{\bigwedge_{m=1}^{N}[(1 - \upsilon^l_{im}) \vee u^j_m]\right\}/x_i,\end{aligned}$$

and the above can be expressed as two type-2 fuzzy sets as follows:

$$\begin{aligned}\overline{R^l_e}(A^j_e) &= \sum_{i=1}^{N}\left\{1/\bigvee_{m=1}^{N}[R^l_e(x_i, y_m) \wedge A^j_e(y_m)]\right\}/x_i \\ &= \sum_{i=1}^{N}\left[1/\bigvee_{m=1}^{N}(\upsilon^l_{im} \wedge u^j_m)\right]/x_i, \\ \underline{R^l_e}(A^j_e) &= \sum_{i=1}^{N}\left\{1/\bigwedge_{m=1}^{N}[(1 - R^l_e(x_i, y_m)) \vee A^j_e(y_m)]\right\}/x_i \\ &= \sum_{i=1}^{N}\left\{1/\bigwedge_{m=1}^{N}[(1 - \upsilon^l_{im}) \vee u^j_m]\right\}/x_i.\end{aligned}$$

In the type-2 case, the Extension Principle can be used to produce the secondary grades, leading to the formulation below:

$$\overline{\tilde{R}^l_e}(\tilde{A}^j_e) = \sum_{i=1}^{N}\left\{\bigwedge_{m=1}^{N}[g_{(x_i,y_m)}(\upsilon^l_{im}) \wedge f_{x_m}(u^j_m)]/\bigvee_{m=1}^{N}(\upsilon^l_{im} \wedge u^j_m)\right\}/x_i, \tag{8}$$

$$\underline{\tilde{R}^l_e}(\tilde{A}^j_e) = \sum_{i=1}^{N}\left\{\bigwedge_{m=1}^{N}[g_{(x_i,y_m)}(\upsilon^l_{im}) \wedge f_{x_m}(u^j_m)]/\bigwedge_{m=1}^{N}[(1 - \upsilon^l_{im}) \vee u^j_m]\right\}/x_i. \tag{9}$$

Then, $\overline{\tilde{R}}(\tilde{A})$ and $\underline{\tilde{R}}(\tilde{A})$ can be deduced from (6) and (7).

Definition 11. Let $\tilde{A}$ be a discrete type-2 fuzzy set on X, and let $\tilde{R}$ be a discrete type-2 fuzzy relation on X, where $X = \{x_1, x_2, \ldots, x_n\}$. If $\tilde{A} = \sum_{j=1}^{n_A}\tilde{A}^j_e$ and $\tilde{R} = \sum_{l=1}^{n_R}\tilde{R}^l_e$, where

$$\tilde{A}^j_e = \sum_{i=1}^{N}[f_{x_i}(u^j_i)/u^j_i]/x_i$$

and

$$\tilde{R}_e^l = \sum_{p,q=1}^{N} [g_{(x_p,y_q)}(v_{pq}^l)/v_{pq}^l]/(x_p, y_q)$$

are the embedded type-2 sets, a type-2 fuzzy rough set is a pair $(\underline{\tilde{R}}(\tilde{A}), \overline{\tilde{R}}(\tilde{A}))$ such that

$$\overline{\tilde{R}}(\tilde{A}) = \sum_{l=1}^{n_R}\sum_{j=1}^{n_A} \overline{\tilde{R}_e^l}(\tilde{A}_e^j),$$

$$\underline{\tilde{R}}(\tilde{A}) = \sum_{l=1}^{n_R}\sum_{j=1}^{n_A} \underline{\tilde{R}_e^l}(\tilde{A}_e^j),$$

where

$$\overline{\tilde{R}_e^l}(\tilde{A}_e^j) = \sum_{i=1}^{N}\left\{ \bigwedge_{m=1}^{N} [g_{(x_i,y_m)}(v_{im}^l) \wedge f_{x_m}(u_m^j)] / \bigvee_{m=1}^{N} (v_{im}^l \wedge u_m^j) \right\}/x_i,$$

$$\underline{\tilde{R}_e^l}(\tilde{A}_e^j) = \sum_{i=1}^{N}\left\{ \bigwedge_{m=1}^{N} [g_{(x_i,y_m)}(v_{im}^l) \wedge f_{x_m}(u_m^j)] / \bigwedge_{m=1}^{N} [(1 - v_{im}^l) \vee u_m^j] \right\}/x_i.$$

The pair $(X, \tilde{R})$ is called a type-2 fuzzy approximation space. The mappings $\underline{\tilde{R}} : \tilde{F}(X) \to \tilde{F}(X)$ and $\overline{\tilde{R}} : \tilde{F}(X) \to \tilde{F}(X)$ are referred to as the lower type-2 fuzzy rough approximation operator and the upper type-2 fuzzy rough approximation operator respectively.

If $\tilde{R}$ is a type-1 fuzzy set, that is, $n_R = 1$ and $\tilde{R} = \sum_{p,q=1}^{N} (1/v_{pq})/(x_p, y_q)$, then:

$$\overline{\tilde{R}}(\tilde{A}) = \sum_{j=1}^{n_A} \overline{\tilde{R}}(\tilde{A}_e^j) = \sum_{j=1}^{n_A}\sum_{i=1}^{N}\left\{ \bigwedge_{m=1}^{N} f_{x_m}(u_m^j) / \bigvee_{m=1}^{N} (v_{im} \wedge u_m^j) \right\}/x_i,$$

$$\underline{\tilde{R}}(\tilde{A}) = \sum_{j=1}^{n_A} \underline{\tilde{R}}(\tilde{A}_e^j) = \sum_{j=1}^{n_A}\sum_{i=1}^{N}\left\{ \bigwedge_{m=1}^{N} f_{x_m}(u_m^j) / \bigwedge_{m=1}^{N} [(1 - v_{im}) \vee u_m^j] \right\}/x_i.$$

Furthermore, if both $\tilde{R}$ and $\tilde{A}$ are type-1 fuzzy sets, that is, $n_R = n_A = 1$ and $\tilde{R} = \sum_{p,q=1}^{N} (1/v_{pq})/(x_p, y_q)$, $\tilde{A} = \sum_{i=1}^{N} (1/u_i)/x_i$, it follows that

$$\overline{\tilde{R}}(\tilde{A}) = \sum_{i=1}^{N}\left\{ 1/ \bigvee_{m=1}^{N} (v_{im} \wedge u_m) \right\}/x_i,$$

$$\underline{\tilde{R}}(\tilde{A}) = \sum_{i=1}^{N}\left\{ 1/ \bigwedge_{m=1}^{N} [(1 - v_{im}) \vee u_m] \right\}/x_i,$$

which is exactly in accordance with the definition of a fuzzy rough set.

Example 3. Fig. 1 depicts a type-2 fuzzy set defined on $X = \{x_1, x_2, x_3\}$:

$$\tilde{A} = (0.8/0.4)/x_1 + (1.0/0.8)/x_1 + (0.6/0.8)/x_2 + (0.5/0.4)/x_3 + (0.3/0.8)/x_3. \tag{10}$$

Note that $M_1^A = 2, M_2^A = 1, M_3^A = 2$, and $n_A = M_1^A M_2^A M_3^A = 4$. Hence, there are four embedded type-2 sets, namely

$$\begin{aligned}
\tilde{A}_e^1 &= (0.8/0.4)/x_1 + (0.6/0.8)/x_2 + (0.5/0.4)/x_3,\\
\tilde{A}_e^2 &= (1.0/0.8)/x_1 + (0.6/0.8)/x_2 + (0.5/0.4)/x_3,\\
\tilde{A}_e^3 &= (0.8/0.4)/x_1 + (0.6/0.8)/x_2 + (0.3/0.8)/x_3,\\
\tilde{A}_e^4 &= (1.0/0.8)/x_1 + (0.6/0.8)/x_2 + (0.3/0.8)/x_3,
\end{aligned} \tag{11}$$

and $\tilde{A} = \sum_{j=1}^{4} \tilde{A}_e^j$.

Consider a type-2 fuzzy relation (Fig. 2) on X:

$$\begin{aligned}
\tilde{R} = {} & (0.5/0.8)/(x_1, x_1) + (0.6/0.4)/(x_1, x_2) + (1.0/0.8)/(x_1, x_2) + (1.0/0.4)/(x_1, x_3) + (0.2/0.8)/(x_2, x_1)\\
& + (0.9/0.8)/(x_2, x_2) + (0.6/0.4)/(x_2, x_3) + (0.1/0.8)/(x_2, x_3) + (0.3/0.8)/(x_3, x_1)\\
& + (0.8/0.4)/(x_3, x_2) + (1.0/0.4)/(x_3, x_3) + (0.5/0.8)/(x_3, x_3).
\end{aligned} \tag{12}$$

There are $n_R = 8$ embedded type-2 sets as follows:

$$\begin{aligned}
\tilde{R}_e^1 = {} & (0.5/0.8)/(x_1, x_1) + (0.6/0.4)/(x_1, x_2) + (1.0/0.4)/(x_1, x_3) + (0.2/0.8)/(x_2, x_1) + (0.9/0.8)/(x_2, x_2)\\
& + (0.6/0.4)/(x_2, x_3) + (0.3/0.8)/(x_3, x_1) + (0.8/0.4)/(x_3, x_2) + (1.0/0.4)/(x_3, x_3),
\end{aligned}$$

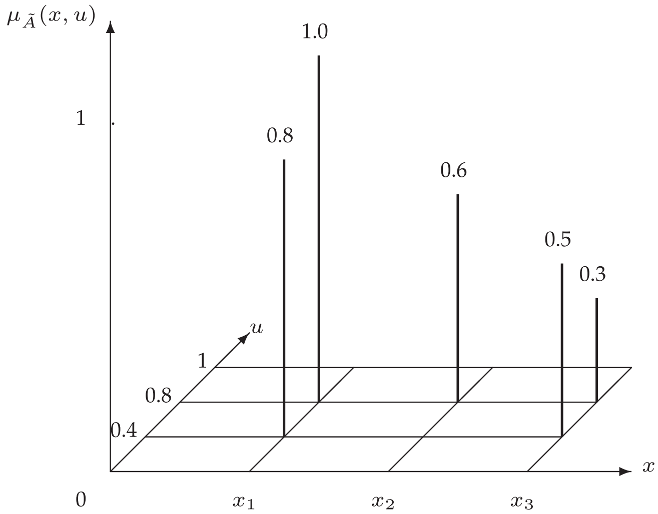

Fig. 1. The type-2 fuzzy set $\tilde{A}$ used in Example 3.

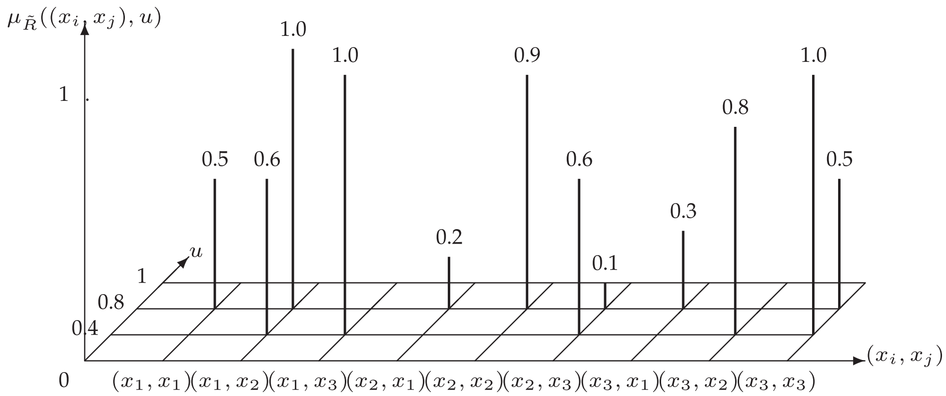

Fig. 2. The type-2 fuzzy relation $\tilde{R}$ used in Example 3.

$$
\begin{aligned}
\tilde{R}_e^2 = {} & (0.5/0.8)/(x_1,x_1) + (1.0/0.8)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.6/0.4)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (1.0/0.4)/(x_3,x_3), \\
\tilde{R}_e^3 = {} & (0.5/0.8)/(x_1,x_1) + (0.6/0.4)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.1/0.8)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (1.0/0.4)/(x_3,x_3), \\
\tilde{R}_e^4 = {} & (0.5/0.8)/(x_1,x_1) + (1.0/0.8)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.1/0.8)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (1.0/0.4)/(x_3,x_3), \\
\tilde{R}_e^5 = {} & (0.5/0.8)/(x_1,x_1) + (0.6/0.4)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.6/0.4)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (0.5/0.8)/(x_3,x_3), \\
\tilde{R}_e^6 = {} & (0.5/0.8)/(x_1,x_1) + (1.0/0.8)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.6/0.4)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (0.5/0.8)/(x_3,x_3), \\
\tilde{R}_e^7 = {} & (0.5/0.8)/(x_1,x_1) + (0.6/0.4)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3) \\
& + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.1/0.8)/(x_2,x_3) \\
& + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (0.5/0.8)/(x_3,x_3),
\end{aligned}
$$

$$\begin{aligned}\tilde{R}_e^8 &= (0.5/0.8)/(x_1,x_1) + (1.0/0.8)/(x_1,x_2) + (1.0/0.4)/(x_1,x_3)\\ &\quad + (0.2/0.8)/(x_2,x_1) + (0.9/0.8)/(x_2,x_2) + (0.1/0.8)/(x_2,x_3)\\ &\quad + (0.3/0.8)/(x_3,x_1) + (0.8/0.4)/(x_3,x_2) + (0.5/0.8)/(x_3,x_3).\end{aligned}$$

Now $\tilde{R} = \sum_{l=1}^{8} \tilde{R}_e^l$ remains to be considered.

$\overline{\tilde{R}_e^l}(\tilde{A}_e^j)$ for $j = 1, 2, 3, 4$ and $l = 1, \ldots, 8$ can be computed by (8), leading to:

$$\begin{aligned}\overline{\tilde{R}}(\tilde{A}) &= \sum_{l=1}^{8}\sum_{j=1}^{4} \overline{\tilde{R}_e^l}(\tilde{A}_e^j)\\ &= (0.5/0.4)/x_1 + (0.5/0.8)/x_1 + (0.2/0.8)/x_2 + (0.3/0.4)/x_3 + (0.3/0.8)/x_3.\end{aligned}$$

Similarly, computing $\underline{\tilde{R}_e^l}(\tilde{A}_e^j)$ for $j = 1, 2, 3, 4$ and $l = 1, \ldots, 8$ using (9) leads to:

$$\begin{aligned}\underline{\tilde{R}}(\tilde{A}) &= \sum_{l=1}^{8}\sum_{j=1}^{4} \underline{\tilde{R}_e^l}(\tilde{A}_e^j)\\ &= (0.5/0.4)/x_1 + (0.5/0.6)/x_1 + (0.3/0.8)/x_1 + (0.2/0.4)/x_2 + (0.2/0.6)/x_2 + (0.2/0.8)/x_2\\ &\quad + (0.3/0.4)/x_3 + (0.3/0.6)/x_3 + (0.3/0.8)/x_3.\end{aligned}$$

4. Granular structure of type-2 fuzzy rough sets

Rough set theory is an important approach to granular computing, but in the fuzzy framework, the granular structure of fuzzy rough sets is not as clear as that of classical rough sets. In 2011, Chen et al. [2] proposed the concept of granular fuzzy sets based on fuzzy similarity relations and described the granular structures of the lower and upper approximations of a fuzzy set. Inspired by [2], in this section, two kinds of granular type-2 fuzzy sets will be defined, and the granular structure of type-2 fuzzy rough sets will be discussed.

Let X be a nonempty universe. A type-2 fuzzy point, denoted by x_λ^μ, is a type-2 fuzzy set defined on X: for any $y \in X$,

$$x_\lambda^\mu(y) = \begin{cases} \mu/\lambda, & \text{if } y = x \\ 1/0, & \text{if } y \neq x \end{cases},$$

where μ/λ (*resp.*, $1/0$) means that at $y = x$ (*resp.*, $y \neq x$), when its primary variable is λ (*resp.*, 0), the associated secondary grade is equal to μ (*resp.*, 1), and at all other values, the secondary grades are equal to 0.

For simplicity, $\tilde{R}_\alpha$ will be used instead of $\tilde{R}_e^j$ to denote the embedded type-2 set described below.

Definition 12. Suppose that $\tilde{R}$ is a type-2 fuzzy relation on X, $\tilde{R}_\alpha$ is an embedded type-2 set of $\tilde{R}$, and R_α is the corresponding embedded type-1 set. For a type-2 fuzzy point x_λ^μ, two granular type-2 fuzzy sets $[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}$ and $[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee}$ are defined as follows: for any $y \in X$,

$$[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}(y) = \frac{\tilde{R}_\alpha((x,y), R_\alpha(x,y)) \wedge \mu}{R_\alpha(x,y) \wedge \lambda}$$

and

$$[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee}(y) = \frac{\tilde{R}_\alpha((x,y), R_\alpha(x,y)) \wedge \mu}{(1 - R_\alpha(x,y)) \vee (1 - \lambda)}.$$

For simplicity, $R_\alpha^*(x,y)$ is used to denote $\tilde{R}_\alpha((x,y), R_\alpha(x,y))$. Then the above equations can be rewritten as:

$$[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}(y) = \frac{R_\alpha^*(x,y) \wedge \mu}{R_\alpha(x,y) \wedge \lambda}$$

and

$$[x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee}(y) = \frac{R_\alpha^*(x,y) \wedge \mu}{(1 - R_\alpha(x,y)) \vee (1 - \lambda)}.$$

In other words, $R_\alpha(x,y)$ is the element chosen from the primary membership of (x,y) by R_α, and $R_\alpha^*(x,y)$ is the associated secondary grade of $R_\alpha(x,y)$.

$[x]_{\tilde{R}_\alpha}^{\wedge}$ and $[x]_{\tilde{R}_\alpha}^{\vee}$ will be used to denote $[x_1^1]_{\tilde{R}_\alpha}^{\wedge}$ and $[x_1^1]_{\tilde{R}_\alpha}^{\vee}$ respectively.

Two type-2 fuzzy sets $[x_\lambda^\mu]_{\tilde{R}}^{\wedge}$ and $[x_\lambda^\mu]_{\tilde{R}}^{\vee}$ can be defined as follows: for any $y \in X$,

$$[x_\lambda^\mu]_{\tilde{R}}^{\wedge}(y) = \sum_\alpha [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}(y), \quad [x_\lambda^\mu]_{\tilde{R}}^{\vee}(y) = \sum_\alpha [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee}(y).$$

Let $M^{\wedge}_{\tilde{R}_\alpha} = \{[x^{\mu}_{\lambda}]^{\wedge}_{\tilde{R}_\alpha} | x \in X; \lambda \in [0,1], \mu \in (0,1]\}$, $M^{\vee}_{\tilde{R}_\alpha} = \{[x^{\mu}_{\lambda}]^{\vee}_{\tilde{R}_\alpha} | x \in X; \lambda \in [0,1], \mu \in (0,1]\}$; $M^{\wedge}_{\tilde{R}} = \{[x^{\mu}_{\lambda}]^{\wedge}_{\tilde{R}} | x \in X; \lambda \in [0,1], \mu \in (0,1]\}$, $M^{\vee}_{\tilde{R}} = \{[x^{\mu}_{\lambda}]^{\vee}_{\tilde{R}} | x \in X; \lambda \in [0,1], \mu \in (0,1]\}$.

Clearly,

$$[x]^{\wedge}_{\tilde{R}}(y) = \sum_{\alpha} [x]^{\wedge}_{\tilde{R}_\alpha}(y) = \sum_{\alpha} \frac{R^*_\alpha(x,y)}{R_\alpha(x,y)} = \tilde{R}(x,y)$$

and

$$[x]^{\vee}_{\tilde{R}}(y) = \sum_{\alpha} [x]^{\vee}_{\tilde{R}_\alpha}(y) = \sum_{\alpha} \frac{R^*_\alpha(x,y)}{1 - R_\alpha(x,y)} = \tilde{R}^c(x,y).$$

Example 4. Consider the type-2 fuzzy relation defined in Example 3 with notations verified: $X = \{a, b, c\}$,

$$\begin{aligned}\tilde{R} = {} & (0.5/0.8)/(a,a) + (0.6/0.4)/(a,b) + (1.0/0.8)/(a,b) + (1.0/0.4)/(a,c) + (0.2/0.8)/(b,a) + (0.9/0.8)/(b,b) \\ & + (0.6/0.4)/(b,c) + (0.1/0.8)/(b,c) + (0.3/0.8)/(c,a) + (0.8/0.4)/(c,b) + (1.0/0.4)/(c,c) + (0.5/0.8)/(c,c).\end{aligned} \tag{13}$$

$$\begin{aligned}\tilde{R}_1 = {} & (0.5/0.8)/(a,a) + (0.6/0.4)/(a,b) + (1.0/0.4)/(a,c) + (0.2/0.8)/(b,a) + (0.9/0.8)/(b,b + (0.6/0.4)/(b,c) \\ & + (0.3/0.8)/(c,a) + (0.8/0.4)/(c,b) + (1.0/0.4)/(c,c)\end{aligned} \tag{14}$$

is an embedded type-2 set of $\tilde{R}$. Take a type-2 fuzzy point $a^{0.5}_1$. It follows that:

$$\begin{aligned}[a^{0.5}_1]^{\wedge}_{\tilde{R}_1} &= \frac{(R^*_1(a,a) \wedge 0.5)/R_1(a,a)}{a} + \frac{(R^*_1(a,b) \wedge 0.5)/R_1(a,b)}{b} + \frac{(R^*_1(a,c) \wedge 0.5)/R_1(a,c)}{c} \\ &= \frac{0.5/0.8}{a} + \frac{0.5/0.4}{b} + \frac{0.5/0.4}{c}\end{aligned}$$

and

$$\begin{aligned}[a^{0.5}_1]^{\vee}_{\tilde{R}_1} &= \frac{(R^*_1(a,a) \wedge 0.5)/(1 - R_1(a,a))}{a} + \frac{(R^*_1(a,b) \wedge 0.5)/(1 - R_1(a,b))}{b} + \frac{(R^*_1(a,c) \wedge 0.5)/(1 - R_1(a,c))}{c} \\ &= \frac{0.5/0.2}{a} + \frac{0.5/0.6}{b} + \frac{0.5/0.6}{c}.\end{aligned}$$

Similarly,

$$\begin{aligned}
[a^{0.5}_1]^{\wedge}_{\tilde{R}_2} &= \frac{0.5/0.8}{a} + \frac{0.5/0.8}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_2} &= \frac{0.5/0.2}{a} + \frac{0.5/0.2}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_3} &= \frac{0.5/0.8}{a} + \frac{0.5/0.4}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_3} &= \frac{0.5/0.2}{a} + \frac{0.5/0.6}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_4} &= \frac{0.5/0.8}{a} + \frac{0.5/0.8}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_4} &= \frac{0.5/0.2}{a} + \frac{0.5/0.2}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_5} &= \frac{0.5/0.8}{a} + \frac{0.5/0.4}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_5} &= \frac{0.5/0.2}{a} + \frac{0.5/0.6}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_6} &= \frac{0.5/0.8}{a} + \frac{0.5/0.8}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_6} &= \frac{0.5/0.2}{a} + \frac{0.5/0.2}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_7} &= \frac{0.5/0.8}{a} + \frac{0.5/0.4}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_7} &= \frac{0.5/0.2}{a} + \frac{0.5/0.6}{b} + \frac{0.5/0.6}{c}, \\
[a^{0.5}_1]^{\wedge}_{\tilde{R}_8} &= \frac{0.5/0.8}{a} + \frac{0.5/0.8}{b} + \frac{0.5/0.4}{c}, & [a^{0.5}_1]^{\vee}_{\tilde{R}_8} &= \frac{0.5/0.2}{a} + \frac{0.5/0.2}{b} + \frac{0.5/0.6}{c}.
\end{aligned}$$

Thus,

$$[a^{0.5}_1]^{\wedge}_{\tilde{R}} = \sum_{\alpha=1}^{8} [a^{0.5}_1]^{\wedge}_{\tilde{R}_\alpha} = \frac{0.5/0.8}{a} + \frac{0.5/0.4 + 0.5/0.8}{b} + \frac{0.5/0.4}{c},$$

$$[a^{0.5}_1]^{\vee}_{\tilde{R}} = \sum_{\alpha=1}^{8} [a^{0.5}_1]^{\vee}_{\tilde{R}_\alpha} = \frac{0.5/0.2}{a} + \frac{0.5/0.6 + 0.5/0.2}{b} + \frac{0.5/0.6}{c}.$$

Next, the properties of granular type-2 fuzzy sets will be discussed.

Theorem 3. *Suppose that $\tilde{R}$ is a type-2 fuzzy relation on X and that $\tilde{R}_\alpha$ is an embedded type-2 set of $\tilde{R}$. For $x \in X$, $\lambda_1, \lambda_2 \in [0,1]$, $\mu_1, \mu_2 \in (0,1]$,*

(1) $[x^{\mu_1}_{\lambda_1}]^{\wedge}_{\tilde{R}_\alpha} \cup [x^{\mu_2}_{\lambda_2}]^{\wedge}_{\tilde{R}_\alpha} = [x^{\mu_1 \wedge \mu_2}_{\lambda_1 \vee \lambda_2}]^{\wedge}_{\tilde{R}_\alpha}$;

(2) $[x_{\lambda_1}^{\mu_1}]_{\tilde{R}_\alpha}^{\wedge} \cap [x_{\lambda_2}^{\mu_2}]_{\tilde{R}_\alpha}^{\wedge} = [x_{\lambda_1\wedge\lambda_2}^{\mu_1\wedge\mu_2}]_{\tilde{R}_\alpha}^{\wedge}$;
(3) $[x_{\lambda_1}^{\mu_1}]_{\tilde{R}_\alpha}^{\vee} \cup [x_{\lambda_2}^{\mu_2}]_{\tilde{R}_\alpha}^{\vee} = [x_{\lambda_1\wedge\lambda_2}^{\mu_1\wedge\mu_2}]_{\tilde{R}_\alpha}^{\vee}$;
(4) $[x_{\lambda_1}^{\mu_1}]_{\tilde{R}_\alpha}^{\vee} \cap [x_{\lambda_2}^{\mu_2}]_{\tilde{R}_\alpha}^{\vee} = [x_{\lambda_1\vee\lambda_2}^{\mu_1\wedge\mu_2}]_{\tilde{R}_\alpha}^{\vee}$.

Proof. (1) For any $y \in X$,

$$\begin{aligned}\left[x_{\lambda_1}^{\mu_1}\right]_{\tilde{R}_\alpha}^{\wedge}(y) \sqcup \left[x_{\lambda_2}^{\mu_2}\right]_{\tilde{R}_\alpha}^{\wedge}(y) &= \frac{R_\alpha^*(x,y)\wedge\mu_1}{R_\alpha(x,y)\wedge\lambda_1} \sqcup \frac{R_\alpha^*(x,y)\wedge\mu_2}{R_\alpha(x,y)\wedge\lambda_2} = \frac{(R_\alpha^*(x,y)\wedge\mu_1)\wedge(R_\alpha^*(x,y)\wedge\mu_2)}{(R_\alpha(x,y)\wedge\lambda_1)\vee(R_\alpha(x,y)\wedge\lambda_2)}\\ &= \frac{R_\alpha^*(x,y)\wedge(\mu_1\wedge\mu_2)}{R_\alpha(x,y)\wedge(\lambda_1\vee\lambda_2)} = \left[x_{\lambda_1\vee\lambda_2}^{\mu_1\wedge\mu_2}\right]_{\tilde{R}_\alpha}^{\wedge}(y).\end{aligned}$$

(2) For any $y \in X$,

$$\begin{aligned}\left[x_{\lambda_1}^{\mu_1}\right]_{\tilde{R}_\alpha}^{\wedge}(y) \sqcap \left[x_{\lambda_2}^{\mu_2}\right]_{\tilde{R}_\alpha}^{\wedge}(y) &= \frac{R_\alpha^*(x,y)\wedge\mu_1}{R_\alpha(x,y)\wedge\lambda_1} \sqcap \frac{R_\alpha^*(x,y)\wedge\mu_2}{R_\alpha(x,y)\wedge\lambda_2} = \frac{(R_\alpha^*(x,y)\wedge\mu_1)\wedge(R_\alpha^*(x,y)\wedge\mu_2)}{(R_\alpha(x,y)\wedge\lambda_1)\wedge(R_\alpha(x,y)\wedge\lambda_2)}\\ &= \frac{R_\alpha^*(x,y)\wedge(\mu_1\wedge\mu_2)}{R_\alpha(x,y)\wedge(\lambda_1\wedge\lambda_2)} = \left[x_{\lambda_1\wedge\lambda_2}^{\mu_1\wedge\mu_2}\right]_{\tilde{R}_\alpha}^{\wedge}(y).\end{aligned}$$

(3) For any $y \in X$,

$$\begin{aligned}\left[x_{\lambda_1}^{\mu_1}\right]_{\tilde{R}_\alpha}^{\vee}(y) \sqcup \left[x_{\lambda_2}^{\mu_2}\right]_{\tilde{R}_\alpha}^{\vee}(y) &= \frac{R_\alpha^*(x,y)\wedge\mu_1}{(1-R_\alpha(x,y))\vee(1-\lambda_1)} \sqcup \frac{R_\alpha^*(x,y)\wedge\mu_2}{(1-R_\alpha(x,y))\vee(1-\lambda_2)}\\ &= \frac{(R_\alpha^*(x,y)\wedge\mu_1)\wedge(R_\alpha^*(x,y)\wedge\mu_2)}{[(1-R_\alpha(x,y))\vee(1-\lambda_1)]\vee[(1-R_\alpha(x,y))\vee(1-\lambda_2)]}\\ &= \frac{R_\alpha^*(x,y)\wedge(\mu_1\wedge\mu_2)}{(1-R_\alpha(x,y))\vee[1-(\lambda_1\wedge\lambda_2)]} = \left[x_{\lambda_1\wedge\lambda_2}^{\mu_1\wedge\mu_2}\right]_{\tilde{R}_\alpha}^{\vee}(y).\end{aligned}$$

(4) For any $y \in X$,

$$\begin{aligned}\left[x_{\lambda_1}^{\mu_1}\right]_{\tilde{R}_\alpha}^{\vee}(y) \sqcap \left[x_{\lambda_2}^{\mu_2}\right]_{\tilde{R}_\alpha}^{\vee}(y) &= \frac{R_\alpha^*(x,y)\wedge\mu_1}{(1-R_\alpha(x,y))\vee(1-\lambda_1)} \sqcap \frac{R_\alpha^*(x,y)\wedge\mu_2}{(1-R_\alpha(x,y))\vee(1-\lambda_2)}\\ &= \frac{(R_\alpha^*(x,y)\wedge\mu_1)\wedge(R_\alpha^*(x,y)\wedge\mu_2)}{[(1-R_\alpha(x,y))\vee(1-\lambda_1)]\wedge[(1-R_\alpha(x,y))\vee(1-\lambda_2)]}\\ &= \frac{R_\alpha^*(x,y)\wedge(\mu_1\wedge\mu_2)}{(1-R_\alpha(x,y))\vee[1-(\lambda_1\vee\lambda_2)]} = \left[x_{\lambda_1\vee\lambda_2}^{\mu_1\wedge\mu_2}\right]_{\tilde{R}_\alpha}^{\vee}(y).\end{aligned}$$

□

Theorem 4. *Suppose that $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$ are type-2 fuzzy relations on X and that $\tilde{R}_\alpha$ and $\tilde{R}_\beta$ are embedded type-2 sets of $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$ respectively. It follows that:*

(1) $[x_\lambda^\mu]_{\tilde{R}_\alpha\cap\tilde{R}_\beta}^{\wedge} = [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge} \cap [x_\lambda^\mu]_{\tilde{R}_\beta}^{\wedge}$;
(2) $[x_\lambda^\mu]_{\tilde{R}_\alpha\cup\tilde{R}_\beta}^{\wedge} = [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge} \cup [x_\lambda^\mu]_{\tilde{R}_\beta}^{\wedge}$;
(3) $[x_\lambda^\mu]_{\tilde{R}_\alpha\cap\tilde{R}_\beta}^{\vee} = [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee} \cup [x_\lambda^\mu]_{\tilde{R}_\beta}^{\vee}$;
(4) $[x_\lambda^\mu]_{\tilde{R}_\alpha\cup\tilde{R}_\beta}^{\vee} = [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\vee} \cap [x_\lambda^\mu]_{\tilde{R}_\beta}^{\vee}$.

Proof. For any $y \in X$,

$$\begin{aligned}[x_\lambda^\mu]_{\tilde{R}_\alpha\cap\tilde{R}_\beta}^{\wedge}(y) &= \frac{[R_\alpha^*(x,y)\wedge R_\beta^*(x,y)]\wedge\mu}{[R_\alpha(x,y)\wedge R_\beta(x,y)]\wedge\lambda}\\ &= \frac{[R_\alpha^*(x,y)\wedge\mu]\wedge[R_\beta^*(x,y)\wedge\mu]}{[R_\alpha(x,y)\wedge\lambda]\wedge[R_\beta(x,y)\wedge\lambda]}\\ &= [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}(y)\sqcap[x_\lambda^\mu]_{\tilde{R}_\beta}^{\wedge}(y).\end{aligned}$$

$$\begin{aligned}[x_\lambda^\mu]_{\tilde{R}_\alpha\cup\tilde{R}_\beta}^{\wedge}(y) &= \frac{[R_\alpha^*(x,y)\wedge R_\beta^*(x,y)]\wedge\mu}{[R_\alpha(x,y)\vee R_\beta(x,y)]\wedge\lambda}\\ &= \frac{[R_\alpha^*(x,y)\wedge\mu]\wedge[R_\beta^*(x,y)\wedge\mu]}{[R_\alpha(x,y)\wedge\lambda]\vee[R_\beta(x,y)\wedge\lambda]}\\ &= [x_\lambda^\mu]_{\tilde{R}_\alpha}^{\wedge}(y)\sqcup[x_\lambda^\mu]_{\tilde{R}_\beta}^{\wedge}(y).\end{aligned}$$

$$
\begin{aligned}
[x_\lambda^\mu]^\vee_{\tilde{R}_\alpha \cap \tilde{R}_\beta}(y) &= \frac{[R^*_\alpha(x,y) \wedge R^*_\beta(x,y)] \wedge \mu}{[1 - R_\alpha(x,y) \wedge R_\beta(x,y)] \vee (1-\lambda)} \\
&= \frac{[R^*_\alpha(x,y) \wedge \mu] \wedge [R^*_\beta(x,y) \wedge \mu]}{[(1 - R_\alpha(x,y)) \vee (1 - R_\beta(x,y))] \vee (1-\lambda)} \\
&= \frac{[R^*_\alpha(x,y) \wedge \mu] \wedge [R^*_\beta(x,y) \wedge \mu]}{[(1 - R_\alpha(x,y)) \vee (1-\lambda)] \vee [(1 - R_\beta(x,y)) \vee (1-\lambda)]} \\
&= [x_\lambda^\mu]^\vee_{\tilde{R}_\alpha}(y) \sqcup [x_\lambda^\mu]^\vee_{\tilde{R}_\beta}(y).
\end{aligned}
$$

$$
\begin{aligned}
[x_\lambda^\mu]^\vee_{\tilde{R}_\alpha \cup \tilde{R}_\beta}(y) &= \frac{[R^*_\alpha(x,y) \wedge R^*_\beta(x,y)] \wedge \mu}{[1 - R_\alpha(x,y) \vee R_\beta(x,y)] \vee (1-\lambda)} \\
&= \frac{[R^*_\alpha(x,y) \wedge \mu] \wedge [R^*_\beta(x,y) \wedge \mu]}{[(1 - R_\alpha(x,y)) \wedge (1 - R_\beta(x,y))] \vee (1-\lambda)} \\
&= \frac{[R^*_\alpha(x,y) \wedge \mu] \wedge [R^*_\beta(x,y) \wedge \mu]}{[(1 - R_\alpha(x,y)) \vee (1-\lambda)] \wedge [(1 - R_\beta(x,y)) \vee (1-\lambda)]} \\
&= [x_\lambda^\mu]^\vee_{\tilde{R}_\alpha}(y) \sqcap [x_\lambda^\mu]^\vee_{\tilde{R}_\beta}(y).
\end{aligned}
$$

□

Theorem 5. *Elements in $M^\wedge_{\tilde{R}_\alpha}$ and $M^\vee_{\tilde{R}_\alpha}$ have the following properties:*

(1) *For every $x \in X$ and $\mu \in (0,1]$, $\lambda' < \lambda$ implies that $[x_{\lambda'}^\mu]^\wedge_{\tilde{R}_\alpha} \subseteq [x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}$ and $[x_{\lambda'}^\mu]^\vee_{\tilde{R}_\alpha} \supseteq [x_\lambda^\mu]^\vee_{\tilde{R}_\alpha}$;*
(2) *For every $x \in X$ and $\lambda \in [0,1]$, $\mu \in (0,1]$, $([x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha})^c = [x_\lambda^\mu]^\vee_{\tilde{R}_\alpha}$, $([x_\lambda^\mu]^\vee_{\tilde{R}_\alpha})^c = [x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}$.*

Proof. (1) The conclusions are obvious.
(2) For every $x \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$,

$$
\begin{aligned}
([x_\lambda^\mu]^\wedge_{R_\alpha})^c(y) &= \frac{\tilde{R}_\alpha(x,y) \wedge \mu}{(1 - R_\alpha(x,y)) \vee (1-\lambda)} = [x_\lambda^\mu]^\vee_{R_\alpha}(y), \\
([x_\lambda^\mu]^\vee_{R_\alpha})^c(y) &= \frac{\tilde{R}_\alpha(x,y) \wedge \mu}{R_\alpha(x,y) \wedge \lambda} = [x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}(y).
\end{aligned}
$$

□

Theorem 6. *Let X be a nonempty universe, and let $\tilde{R}$ be a type-2 fuzzy relation on X. Then the following statements are equivalent:*

(1) *$\tilde{R}$ is reflexive*;
(2) *For any $x \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$, $[x_\lambda^\mu]^\wedge_{\tilde{R}}(x) = \mu/\lambda$;*
(3) *For any $x \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$, $[x_\lambda^\mu]^\vee_{\tilde{R}}(x) = \mu/(1-\lambda)$.*

Proof. (1) ⇒ (3): Because for every $x \in X$, $\tilde{R}(x,x) = 1/1$, then for any embedded type-2 set $\tilde{R}_\alpha$, $\tilde{R}_\alpha(x,x) = 1/1$. Hence,

$$[x_\lambda^\mu]^\vee_{\tilde{R}}(x) = \sum_\alpha [x_\lambda^\mu]^\vee_{\tilde{R}_\alpha}(x) = \sum_\alpha \frac{\tilde{R}_\alpha(x,x) \wedge \mu}{(1 - R_\alpha(x,x)) \vee (1-\lambda)} = \mu/(1-\lambda).$$

(3) ⇒ (2):

$$[x_\lambda^\mu]^\wedge_{\tilde{R}}(x) = \sum_\alpha [x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}(x) = \sum_\alpha \frac{\tilde{R}_\alpha(x,x) \wedge \mu}{R_\alpha(x,x) \wedge \lambda} = \neg [x_\lambda^\mu]^\vee_{\tilde{R}}(x) = \mu/\lambda.$$

(2) ⇒ (1): Let $x \in X$, if there exists $\tilde{R}_\alpha$ such that $R_\alpha(x,x) = \lambda' < 1$ and $\tilde{R}_\alpha(x,x) > 0$. Suppose that $\lambda > \lambda'$; then, $[x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}(x) = \frac{\tilde{R}_\alpha(x,x) \wedge \mu}{R_\alpha(x,x) \wedge \lambda} \neq \frac{\mu}{\lambda}$, which contradicts assertion (2). Therefore, for any $x \in X$ and for any $\lambda < 1$, $\tilde{R}((x,x),\lambda) = 0$. Moreover, if there exists an $x \in X$, such that $\tilde{R}(x,x) = \mu'/1$ with $\mu' < 1$, then for $\mu > \mu'$, $[x_\lambda^\mu]^\wedge_{\tilde{R}}(x) = \sum_\alpha [x_\lambda^\mu]^\wedge_{\tilde{R}_\alpha}(x) = \sum_\alpha \frac{\tilde{R}_\alpha(x,x) \wedge \mu}{R_\alpha(x,x) \wedge \lambda} = \mu'/\lambda \neq \mu/\lambda$, which is also a contradiction. In conclusion, $\tilde{R}(x,x) = 1/1$. □

Theorem 7. *Let X be a nonempty universe, and let $\tilde{R}$ be a type-2 fuzzy relation on X. Then the following statements are equivalent:*

(1) *$\tilde{R}$ is symmetric*;
(2) *For any $x, y \in X$, $[x]^\wedge_{\tilde{R}}(y) = [y]^\wedge_{\tilde{R}}(x)$;*
(3) *For any $x, y \in X$, $[x]^\vee_{\tilde{R}}(y) = [y]^\vee_{\tilde{R}}(x)$.*

Proof. $\tilde{R}$ is symmetric if and only if $\tilde{R}(x,y) = \tilde{R}(y,x)$, which is equivalent to $\tilde{R}^c(x,y) = \tilde{R}^c(y,x)$, and therefore the above theorem holds. □

Theorem 8. *Let X be a nonempty universe, and let $\tilde{R}$ be a type-2 fuzzy relation on X. Then the following statements are equivalent:*

(1) *$\tilde{R}$ is transitive*;
(2) *For any $x, y \in X$, $[x]^{\wedge}_{\tilde{R}}(y) \geq \sqcup_{z\in X}([x]^{\wedge}_{\tilde{R}}(z) \sqcap [z]^{\wedge}_{\tilde{R}}(y))$*;
(3) *For any $x, y \in X$, $([x]^{\vee}_{\tilde{R}})^c(y) \geq \sqcup_{z\in X}(([x]^{\vee}_{\tilde{R}})^c(z) \sqcap ([z]^{\vee}_{\tilde{R}})^c(y))$.*

Proof. These conclusions follow from the facts that $[x]^{\wedge}_{\tilde{R}}(y) = \tilde{R}(x,y)$ and $([x]^{\vee}_{\tilde{R}})^c = [x]^{\wedge}_{\tilde{R}}$. □

Different from the property of fuzzy *T*-transitive relation proven by Chen et al. ([2], Proposition 3.1.5), the last property cannot be replaced by $[x]^{\vee}_{\tilde{R}}(y) \leq \sqcap_{z\in X}([x]^{\vee}_{\tilde{R}}(z) \sqcup [z]^{\vee}_{\tilde{R}}(y))$.

Example 5. Consider the type-2 fuzzy similarity relation $\tilde{R}$ defined in Example 2, $[a]^{\vee}_{\tilde{R}}(b) = \tilde{R}^c(a,b) = 1/0.2$, and

$$\sqcap_{z\in X}([a]^{\vee}_{\tilde{R}}(z) \sqcup [z]^{\vee}_{\tilde{R}}(b)) = \sqcap_{z\in X}(\tilde{R}^c(a,z) \sqcup \tilde{R}^c(z,b)) = 0.5/0.2.$$

It is clear that $[a]^{\vee}_{\tilde{R}}(b) \not\leq \sqcap_{z\in X}([a]^{\vee}_{\tilde{R}}(z) \sqcup [z]^{\vee}_{\tilde{R}}(b))$.

Theorem 9. *Let X be a nonempty universe, and let $\tilde{R}$ be a type-2 fuzzy relation on X. Then the following statements are equivalent:*

(1) *$\tilde{R}$ is a type-2 fuzzy similarity relation*;
(2) *For any $x, y \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$, $[x^{\mu}_{\lambda}]^{\wedge}_{\tilde{R}}(x) = \mu/\lambda$, $[x]^{\wedge}_{\tilde{R}}(y) = [y]^{\wedge}_{\tilde{R}}(x)$, $[x]^{\wedge}_{\tilde{R}}(y) \geq \sqcup_{z\in X}([x]^{\wedge}_{\tilde{R}}(z) \sqcap [z]^{\wedge}_{\tilde{R}}(y))$*;
(3) *For any $x, y \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$, $[x^{\mu}_{\lambda}]^{\vee}_{\tilde{R}}(x) = \mu/(1-\lambda)$, $[x]^{\vee}_{\tilde{R}}(y) = [y]^{\vee}_{\tilde{R}}(x)$, $([x]^{\vee}_{\tilde{R}})^c(y) \geq \sqcup_{z\in X}(([x]^{\vee}_{\tilde{R}})^c(z) \sqcap ([z]^{\vee}_{\tilde{R}})^c(y))$.*

Proof. Combining Theorem 6, Theorem 7 and Theorem 8, we have the theorem. □

Different from the property of fuzzy similarity relation proven by Chen et al. ([2], Theorem 3.1.6), the inequalities in (2) and (3) cannot be replaced by equalities because type-2 fuzzy sets do not satisfy absorption laws [14].

Example 6. Consider the type-2 fuzzy similarity relation $\tilde{R}$ given in Example 2, $[a]^{\wedge}_{\tilde{R}}(b) = ([a]^{\vee}_{\tilde{R}})^c(b) = \tilde{R}(a,b) = 1/0.8$, and $\sqcup_{z\in X}([a]^{\wedge}_{\tilde{R}}(z) \sqcap [z]^{\wedge}_{\tilde{R}}(b)) = \sqcup_{z\in X}(([a]^{\vee}_{\tilde{R}})^c(z) \sqcap ([z]^{\vee}_{\tilde{R}})^c(b)) = \sqcup_{z\in X}(\tilde{R}(a,z) \sqcap \tilde{R}(z,b)) = 0.5/0.8$. Clearly, $[a]^{\wedge}_{\tilde{R}}(b) \neq \sqcup_{z\in X}([a]^{\wedge}_{\tilde{R}}(z) \sqcap [z]^{\wedge}_{\tilde{R}}(b))$ and $([a]^{\vee}_{\tilde{R}})^c(b) \neq \sqcup_{z\in X}(([a]^{\vee}_{\tilde{R}})^c(z) \sqcap ([z]^{\vee}_{\tilde{R}})^c(b))$.

Suppose that $\tilde{R}$ is a type-2 fuzzy relation on *U*. Define two mappings

$$\Gamma^{\wedge}_{\tilde{R}} : \{x^{\mu}_{\lambda} : x \in X, \lambda \in [0,1], \mu \in (0,1]\} \to M^{\wedge}_{\tilde{R}}$$

and

$$\Gamma^{\vee}_{\tilde{R}} : \{x^{\mu}_{\lambda} : x \in X, \lambda \in [0,1], \mu \in (0,1]\} \to M^{\vee}_{\tilde{R}}$$

as follows: for any x^{μ}_{λ},

$$\Gamma^{\wedge}_{\tilde{R}}(x^{\mu}_{\lambda}) = [x^{\mu}_{\lambda}]^{\wedge}_{\tilde{R}} \text{ and } \Gamma^{\vee}_{\tilde{R}}(x^{\mu}_{\lambda}) = [x^{\mu}_{\lambda}]^{\vee}_{\tilde{R}}.$$

Theorem 10. *Suppose that $\Gamma^{\wedge}, \Gamma^{\vee} : \{x^{\mu}_{\lambda} : x \in X, \lambda \in [0,1], \mu \in (0,1]\} \to \tilde{F}(X)$ are two mappings. Then there exists a type-2 fuzzy relation $\tilde{R}$ such that $\Gamma^{\wedge} = \Gamma^{\wedge}_{\tilde{R}}$ and $\Gamma^{\vee} = \Gamma^{\vee}_{\tilde{R}}$ if and only if $\Gamma^{\wedge}$ and $\Gamma^{\vee}$ satisfy the following axioms: For any $x, y \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$,*

(1) $\Gamma^{\wedge}(x^{\mu}_{\lambda})(y) = \Gamma^{\wedge}(x^1_1)(y) \sqcap \mu/\lambda$, $\Gamma^{\vee}(x^{\mu}_{\lambda})(y) = \Gamma^{\vee}(x^1_1)(y) \sqcup \mu/(1-\lambda)$.
(2) $(\Gamma^{\wedge}(x^{\mu}_{\lambda}))^c = \Gamma^{\vee}(x^{\mu}_{\lambda})$, $(\Gamma^{\vee}(x^{\mu}_{\lambda}))^c = \Gamma^{\wedge}(x^{\mu}_{\lambda})$.

Proof. Necessity. For the first axiom:

$$\begin{aligned}\Gamma^{\wedge}(x^1_1)(y) \sqcap \frac{\mu}{\lambda} &= \Gamma^{\wedge}_{\tilde{R}}(x^1_1)(y) \sqcap \frac{\mu}{\lambda} = [x^1_1]^{\wedge}_{\tilde{R}}(y) \sqcap \frac{\mu}{\lambda} \\ &= \sum_{\alpha} \frac{R^*_{\alpha}(x,y)}{R_{\alpha}(x,y)} \sqcap \frac{\mu}{\lambda} = \sum_{\alpha} \frac{R^*_{\alpha}(x,y) \wedge \mu}{R_{\alpha}(x,y) \wedge \lambda} \\ &= [x^{\mu}_{\lambda}]^{\wedge}_{\tilde{R}}(y) = \Gamma^{\wedge}_{\tilde{R}}(x^{\mu}_{\lambda})(y) = \Gamma^{\wedge}(x^{\mu}_{\lambda})(y);\end{aligned}$$

$$\begin{aligned}\Gamma^{\vee}(x^1_1)(y) \sqcup \frac{\mu}{1-\lambda} &= \Gamma^{\vee}_{\tilde{R}}(x^1_1)(y) \sqcup \frac{\mu}{1-\lambda} = [x^1_1]^{\vee}_{\tilde{R}}(y) \sqcup \frac{\mu}{1-\lambda} \\ &= \sum_{\alpha} \frac{R^*_{\alpha}(x,y)}{1-R_{\alpha}(x,y)} \sqcup \frac{\mu}{1-\lambda}\end{aligned}$$

$$= \sum_{\alpha} \frac{R^*_\alpha(x,y) \wedge \mu}{(1 - R_\alpha(x,y)) \vee (1-\lambda)}$$
$$= [x^\mu_\lambda]^\vee_{\tilde{R}}(y) = \Gamma^\vee_{\tilde{R}}(x^\mu_\lambda)(y) = \Gamma^\vee(x^\mu_\lambda)(y).$$

For the second axiom:

$$\begin{aligned}(\Gamma^\wedge(x^\mu_\lambda))^c(y) &= (\Gamma^\wedge_{\tilde{R}}(x^\mu_\lambda))^c(y) = ([x^\mu_\lambda]^\wedge_{\tilde{R}})^c(y)\\ &= \neg\left(\sum_\alpha \frac{R^*_\alpha(x,y)\wedge\mu}{R_\alpha(x,y)\wedge\lambda}\right)\\ &= \sum_\alpha \frac{R^*_\alpha(x,y)\wedge\mu}{(1-R_\alpha(x,y))\vee(1-\lambda)}\\ &= [x^\mu_\lambda]^\vee_{\tilde{R}}(y) = \Gamma^\vee_{\tilde{R}}(x^\mu_\lambda)(y) = \Gamma^\vee(x^\mu_\lambda)(y);\end{aligned}$$

$$\begin{aligned}(\Gamma^\vee(x^\mu_\lambda))^c(y) &= (\Gamma^\vee_{\tilde{R}}(x^\mu_\lambda))^c(y) = ([x^\mu_\lambda]^\vee_{\tilde{R}})^c(y)\\ &= \neg\left(\sum_\alpha \frac{R^*_\alpha(x,y)\wedge\mu}{(1-R_\alpha(x,y))\vee(1-\lambda)}\right)\\ &= \sum_\alpha \frac{R^*_\alpha(x,y)\wedge\mu}{R_\alpha(x,y)\wedge\lambda}\\ &= [x^\mu_\lambda]^\wedge_{\tilde{R}}(y) = \Gamma^\wedge_{\tilde{R}}(x^\mu_\lambda)(y) = \Gamma^\wedge(x^\mu_\lambda)(y).\end{aligned}$$

Sufficiency. Suppose that $\Gamma^\wedge$ and $\Gamma^\vee$ satisfy axioms (1) and (2). Define two type-2 fuzzy relations on X as $\tilde{R}(x,y) = \Gamma^\wedge(x^1_1)(y)$ and $\tilde{P}(x,y) = \neg(\Gamma^\vee(x^1_1)(y))$. It follows that:

$$\begin{aligned}\Gamma^\wedge(x^\mu_\lambda)(y) &= \Gamma^\wedge(x^1_1)(y) \sqcap \frac{\mu}{\lambda} = \tilde{R}(x,y) \sqcap \frac{\mu}{\lambda}\\ &= \sum_\alpha \frac{R^*_\alpha(x,y)}{R_\alpha(x,y)} \sqcap \frac{\mu}{\lambda}\\ &= \sum_\alpha \frac{R^*_\alpha(x,y)\wedge\mu}{R_\alpha(x,y)\wedge\lambda}\\ &= [x^\mu_\lambda]^\wedge_{\tilde{R}}(y) = \Gamma^\wedge_{\tilde{R}}(x^\mu_\lambda)(y);\\ \Gamma^\vee(x^\mu_\lambda)(y) &= \Gamma^\vee(x^1_1)(y) \sqcup \frac{\mu}{1-\lambda} = \neg\tilde{P}(x,y) \sqcup \frac{\mu}{1-\lambda}\\ &= \sum_\alpha \frac{P^*_\alpha(x,y)}{1-P_\alpha(x,y)} \sqcup \frac{\mu}{1-\lambda}\\ &= \sum_\alpha \frac{P^*_\alpha(x,y)\wedge\mu}{(1-P_\alpha(x,y))\vee(1-\lambda)}\\ &= [x^\mu_\lambda]^\vee_{\tilde{P}}(y) = \Gamma^\vee_{\tilde{P}}(x^\mu_\lambda)(y).\end{aligned}$$

Hence, $\Gamma^\wedge = \Gamma^\wedge_{\tilde{R}}$ and $\Gamma^\vee = \Gamma^\vee_{\tilde{P}}$. Next, it will be proven that $\tilde{R} = \tilde{P}$. Actually, $\tilde{R}(x,y) = \Gamma^\wedge(x^1_1)(y) = (\Gamma^\vee(x^1_1))^c(y) = \neg(\Gamma^\vee(x^1_1)(y)) = \tilde{P}(x,y)$. So $\tilde{R} = \tilde{P}$. □

Theorem 11. *Suppose that $\Gamma^\wedge, \Gamma^\vee : \{x^\mu_\lambda : x \in X, \lambda \in [0,1], \mu \in (0,1]\} \to \tilde{F}(X)$ are two mappings satisfying the axioms in the above theorem. Then there exists a type-2 fuzzy similarity relation $\tilde{R}$ such that $\Gamma^\wedge = \Gamma^\wedge_{\tilde{R}}$ and $\Gamma^\vee = \Gamma^\vee_{\tilde{R}}$ if and only if $\Gamma^\wedge$ and $\Gamma^\vee$ satisfy the following axioms: For any $x, y \in X$, $\lambda \in [0,1]$, $\mu \in (0,1]$,*

(1) $\Gamma^\wedge(x^\mu_\lambda)(x) = \frac{\mu}{\lambda}$, $\Gamma^\wedge(x^1_1)(y) = \Gamma^\wedge(y^1_1)(x)$, $\Gamma^\wedge(x^1_1)(y) \geq \sqcup_{z\in X}(\Gamma^\wedge(x^1_1)(z) \sqcap \Gamma^\wedge(z^1_1)(y))$;
(2) $\Gamma^\vee(x^\mu_\lambda)(x) = \frac{\mu}{1-\lambda}$, $\Gamma^\vee(x^1_1)(y) = \Gamma^\vee(y^1_1)(x)$, $(\Gamma^\vee(x^1_1))^c(y) \geq \sqcup_{z\in X}[(\Gamma^\vee(x^1_1))^c(z) \sqcap (\Gamma^\vee(z^1_1))^c(y)]$.

Proof. Sufficiency. Let $\tilde{R}(x,y) = \Gamma^\wedge(x^1_1)(y)$, then $\Gamma^\wedge = \Gamma^\wedge_{\tilde{R}}$ and $\Gamma^\vee = \Gamma^\vee_{\tilde{R}}$. If $\Gamma^\wedge$ satisfies axiom (1) or $\Gamma^\vee$ satisfies axiom (2), it is obvious that $\tilde{R}$ is a type-2 fuzzy similarity relation. Necessity is obvious. □

Next, the granular structures of the lower and upper approximations of a type-2 fuzzy set will be described.

Theorem 12. *Let $\tilde{R}$ be a discrete type-2 fuzzy relation on X, and let $\tilde{A}$ be a discrete type-2 fuzzy set on X, where X is a nonempty universe. If these can be represented by the Representation Theorem as $\tilde{R} = \sum_{\alpha=1}^{n_R} \tilde{R}_\alpha$ and $\tilde{A} = \sum_{\beta=1}^{n_A} \tilde{A}_\beta$, then*

$$\overline{\tilde{R}_\alpha}(\tilde{A}_\beta) = \bigcup_{x\in X}\left[x^{\tilde{A}_\beta(x)}_{A_\beta(x)}\right]^{\wedge}_{\tilde{R}_\alpha}$$

and

$$\underline{\tilde{R}_\alpha}(\tilde{A}_\beta) = \bigcap_{x\in X}\left[x^{\tilde{A}^c_\beta(x)}_{A^c_\beta(x)}\right]^{\vee}_{\tilde{R}_\alpha}.$$

Consequently, the upper and lower approximations of $\tilde{A}$ *can be represented as*

$$\overline{\tilde{R}}(\tilde{A}) = \sum_{\alpha=1}^{n_R}\sum_{\beta=1}^{n_A}\overline{\tilde{R}_\alpha}(\tilde{A}_\beta) = \sum_{\alpha=1}^{n_R}\sum_{\beta=1}^{n_A}\bigcup_{x\in X}\left[x^{\tilde{A}_\beta(x)}_{A_\beta(x)}\right]^{\wedge}_{\tilde{R}_\alpha},$$

$$\underline{\tilde{R}}(\tilde{A}) = \sum_{\alpha=1}^{n_R}\sum_{\beta=1}^{n_A}\underline{\tilde{R}_\alpha}(\tilde{A}_\beta) = \sum_{\alpha=1}^{n_R}\sum_{\beta=1}^{n_A}\bigcap_{x\in X}\left[x^{\tilde{A}^c_\beta(x)}_{A^c_\beta(x)}\right]^{\vee}_{\tilde{R}_\alpha}.$$

Proof.

$$\bigcup_{x\in X}\left[x^{\tilde{A}_\beta(x)}_{A_\beta(x)}\right]^{\wedge}_{\tilde{R}_\alpha}(y) = \bigcup_{x\in X}\frac{R^*_\alpha(x,y)\wedge\tilde{A}_\beta(x)}{R_\alpha(x,y)\wedge A_\beta(x)} = \frac{\bigwedge_{x\in X}(R^*_\alpha(x,y)\wedge\tilde{A}_\beta(x))}{\bigvee_{x\in X}(R_\alpha(x,y)\wedge A_\beta(x))} = \overline{\tilde{R}_\alpha}(\tilde{A}_\beta)(y),$$

$$\bigcap_{x\in X}\left[x^{\tilde{A}^c_\beta(x)}_{A^c_\beta(x)}\right]^{\vee}_{\tilde{R}_\alpha}(y) = \bigcap_{x\in X}\frac{R^*_\alpha(x,y)\wedge\tilde{A}^c_\beta(x)}{(1-R_\alpha(x,y))\vee(1-A^c_\beta(x))} = \frac{\bigwedge_{x\in X}(R^*_\alpha(x,y)\wedge\tilde{A}_\beta(x))}{\bigwedge_{x\in X}[(1-R_\alpha(x,y))\vee A_\beta(x)]} = \underline{\tilde{R}_\alpha}(\tilde{A}_\beta)(y),$$

where A_β is the embedded type-1 set associated with $\tilde{A}_\beta$ and $\tilde{A}_\beta(x)$ is the simplified form of $\tilde{A}_\beta(x, A_\beta(x))$. □

In classical rough set theory, the upper approximation of A related to the equivalence relation R can be explained as the union of the minimal basic granules to which every element in A belongs, where the collection of all equivalence classes is the basic granular set [2], that is, $\overline{R}(A) = \cup\{[x]_R | x \in A\}$. The lower approximation of A is $\underline{R}A = \cap\{([x]_R)^c | x \in A^c\}$. Considering $\bigcup_{\alpha=1}^{n_R} M^{\wedge}_{\tilde{R}_\alpha}$ and $\bigcup_{\alpha=1}^{n_R} M^{\vee}_{\tilde{R}_\alpha}$ as the basic granular set, then the above theorem shows the corresponding results in the type-2 fuzzy frame.

To compare the model presented here with that proposed by Zhao and Xiao [26], let us consider the example in [26] with some notations modified.

Example 7. Let $X = \{x_1, x_2, x_3\}$. $\tilde{A} \in \tilde{F}(X)$ and $\tilde{R} \in \tilde{F}(X \times X)$ are given as follows:
$\mu_{\tilde{A}}(x_1) = trimf(0.18, 0.36, 0.54)$;
$\mu_{\tilde{A}}(x_2) = trapmf(0.38, 0.52, 0.74, 0.86)$;
$\mu_{\tilde{A}}(x_3) = trimf(0.39, 0.61, 0.72)$;
$\mu_{\tilde{R}}(x_1, x_1) = \mu_{\tilde{R}}(x_2, x_2) = \mu_{\tilde{R}}(x_3, x_3) = 1/1$;
$\mu_{\tilde{R}}(x_1, x_2) = \mu_{\tilde{R}}(x_2, x_1) = trapmf(0.2, 0.3, 0.37, 0.45)$;
$\mu_{\tilde{R}}(x_1, x_3) = \mu_{\tilde{R}}(x_3, x_1) = trimf(0.1, 0.23, 0.39)$;
$\mu_{\tilde{R}}(x_2, x_3) = \mu_{\tilde{R}}(x_3, x_2) = trimf(0.8, 0.95, 1)$,

where *trapmf*(·, ·, ·, ·) denotes a trapezoidal function, the first and fourth parameters of () denote the bottom left and right endpoints respectively, and the second and third parameters of () denote the top left and right endpoints respectively. Furthermore, *trimf*(·, ·, ·) denotes a triangular function, the first and third parameters of () denote the bottom left and right endpoints respectively, and the second parameter of () denotes the apex.

Because the model presented here is defined for discrete type-2 fuzzy sets and relations, the secondary membership functions of A and R can be discretized. For purposes of comparison, embedded type-2 sets were chosen as follows:

$$\tilde{A}^1_e = \frac{0.5/0.27}{x_1} + \frac{0.5/0.45}{x_2} + \frac{0.5/0.5}{x_3},$$

$$\tilde{A}^2_e = \frac{0.5/0.45}{x_1} + \frac{0.5/0.8}{x_2} + \frac{0.5/0.66}{x_3},$$

$$\tilde{R}^1_e = \frac{0.5/1}{(x_1,x_1)} + \frac{0.5/0.25}{(x_1,x_2)} + \frac{0.5/0.17}{(x_1,x_3)} + \frac{0.5/0.25}{(x_2,x_1)} + \frac{0.5/1}{(x_2,x_2)} + \frac{0.5/0.88}{(x_2,x_3)} + \frac{0.5/0.17}{(x_3,x_1)} + \frac{0.5/0.88}{(x_3,x_2)} + \frac{0.5/1}{(x_3,x_3)},$$

$$\tilde{R}^2_e = \frac{0.5/1}{(x_1,x_1)} + \frac{0.5/0.41}{(x_1,x_2)} + \frac{0.5/0.31}{(x_1,x_3)} + \frac{0.5/0.41}{(x_2,x_1)} + \frac{0.5/1}{(x_2,x_2)} + \frac{0.5/0.97}{(x_2,x_3)} + \frac{0.5/0.31}{(x_3,x_1)} + \frac{0.5/0.97}{(x_3,x_2)} + \frac{0.5/1}{(x_3,x_3)}.$$

The corresponding embedded type-1 sets of $\tilde{A}_e^1$ and $\tilde{A}_e^2$ are exactly $S_L^A(x|0.5)$ and $S_U^A(x|0.5)$ in [26], and the corresponding embedded type-1 sets of $\tilde{R}_e^1$ and $\tilde{R}_e^2$ are exactly $S_L^R((x,y)|0.5)$ and $S_U^R((x,y)|0.5)$ in [26].

Assume that $\tilde{A}_\beta = \tilde{A}_e^1$ and $\tilde{R}_\alpha = \tilde{R}_e^1$. It follows that:

$$\overline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \cup_{x\in X}\left[x^{\tilde{A}_\beta(x)}_{A_\beta(x)}\right]^{\wedge}_{\tilde{R}_\alpha} = \frac{0.5/0.27}{x_1} + \frac{0.5/0.25}{x_2} + \frac{0.5/0.17}{x_3},$$
$$\underline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \cap_{x\in X}\left[x^{\tilde{A}^c_\beta(x)}_{A^c_\beta(x)}\right]^{\vee}_{\tilde{R}_\alpha} = \frac{0.5/0.27}{x_1} + \frac{0.5/0.45}{x_2} + \frac{0.5/0.45}{x_3}.$$

Now assume that $\tilde{A}_\beta = \tilde{A}_e^2$ and $\tilde{R}_\alpha = \tilde{R}_e^1$. Then,

$$\overline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.45}{x_1} + \frac{0.5/0.8}{x_2} + \frac{0.5/0.8}{x_3},$$
$$\underline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.45}{x_1} + \frac{0.5/0.66}{x_2} + \frac{0.5/0.66}{x_3}.$$

Next, assume that $\tilde{A}_\beta = \tilde{A}_e^1$ and $\tilde{R}_\alpha = \tilde{R}_e^2$. Then,

$$\overline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.41}{x_1} + \frac{0.5/0.5}{x_2} + \frac{0.5/0.5}{x_3},$$
$$\underline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.27}{x_1} + \frac{0.5/0.45}{x_2} + \frac{0.5/0.45}{x_3}.$$

Assuming $\tilde{A}_\beta = \tilde{A}_e^2$ and $\tilde{R}_\alpha = \tilde{R}_e^2$, then

$$\overline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.45}{x_1} + \frac{0.5/0.8}{x_2} + \frac{0.5/0.8}{x_3},$$
$$\underline{\tilde{R}}_\alpha(\tilde{A}_\beta) = \frac{0.5/0.45}{x_1} + \frac{0.5/0.59}{x_2} + \frac{0.5/0.66}{x_3}.$$

In comparison with the results given in [26], it has been found here that the corresponding embedded type-1 set of $\overline{\tilde{R}_e^1}(\tilde{A}_e^1)$ is $S_L^{\overline{f}(A)}(x|0.5)$, the corresponding embedded type-1 set of $\underline{\tilde{R}_e^1}(\tilde{A}_e^2)$ is $S_U^{\underline{f}(A)}(x|0.5)$, the corresponding embedded type-1 set of $\underline{\tilde{R}_e^2}(\tilde{A}_e^1)$ is $S_L^{\underline{f}(A)}(x|0.5)$, and the corresponding embedded type-1 set of $\overline{\tilde{R}_e^2}(\tilde{A}_e^2)$ is $S_U^{\overline{f}(A)}(x|0.5)$.

Similarly, other approximate results can be computed for other embedded type-2 sets and relations. It is clear that the model presented here will result in the same upper and lower approximations of $\tilde{A}$ as that proposed by Zhao and Xiao.

Example 8. As a type-2 fuzzy rough attribute reduction example, consider a type-2 fuzzy decision system $(X, \mathbf{R}\cup\mathbf{D})$. Suppose that $X = \{e_1, e_2, \ldots, e_6\}$ is a set of finite objects , $\mathbf{R} = \{R_1, R_2, \tilde{R}_3\}$ is a set of conditional attributes, and $D = \{Q\}$ is a set of decision attributes. R_1, R_2, Q are crisp equivalence relations, and $\tilde{R}_3$ is a type-2 fuzzy relation. These are defined as follows:

$$R_1(e_i,e_j) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$R_2(e_i,e_j) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\tilde{R}_3(e_i,e_j) = \begin{pmatrix} 1/1 & 1/0.6 & 1/0.2 & 1/0.9 & 1/0.5 & 1/0.1\\ 1/0.6 & 1/1 & 1/0.7 & 1/0.6 & 0.5/0.8 & 1/0.6\\ 1/0.2 & 1/0.7 & 1/1 & 1/0.2 & 1/0.8 & 0.5/0.9\\ 1/0.9 & 1/0.6 & 1/0.2 & 1/1 & 1/0.6 & 1/0.1+1/0.2\\ 1/0.5 & 0.5/0.8 & 1/0.8 & 1/0.6 & 1/1 & 1/0.8\\ 1/0.1 & 1/0.6 & 0.5/0.9 & 1/0.1+1/0.2 & 1/0.8 & 1/1 \end{pmatrix}$$

$$X/Q = \{Y_1, Y_2\} = \{\{e_2, e_3, e_6\}, \{e_1, e_4, e_5\}\}.$$

Table 1
Example of rough attribute reduction.

	a_1	a_2	a_3	d
e1	yes	yes	normal	No
e2	yes	yes	high	yes
e3	yes	yes	very high	Yes
e4	no	yes	normal	no
e5	no	no	high	No
e6	no	yes	very high	yes

Because R_1 and R_2 are crisp equivalence relations and

$$X/R_1 = \{\{e_1, e_2, e_3\}, \{e_4, e_5, e_6\}\},$$
$$X/R_2 = \{\{e_1, e_2, e_3, e_4, e_6\}, \{e_5\}\},$$

it is clear that:

$$\gamma_1(Q) = \frac{|\underline{R_1}(Y_1) \cup \underline{R_1}(Y_2)|}{|X|} = 0,$$
$$\gamma_2(Q) = \frac{|\underline{R_2}(Y_1) \cup \underline{R_2}(Y_2)|}{|X|} = 1/6.$$

Suppose that $\tilde{R}_3 = \sum_\alpha \tilde{R}_\alpha$. Then for any $y \in X$ and $i = 1, 2$,

$$\begin{aligned}\underline{\tilde{R}_3}(Y_i)(y) &= \sum_\alpha \underline{R_\alpha}(Y_i)(y)\\ &= \sum_\alpha \frac{\wedge_{x\in X} R^*_\alpha(x, y)}{\wedge_{x\in X}[(1 - R_\alpha(x, y)) \vee Y_i(x)]}\\ &= \sum_\alpha \frac{\wedge_{x\in X} R^*_\alpha(x, y)}{\wedge_{x\in Y_i}[(1 - R_\alpha(x, y)) \vee Y_i(x)] \wedge \{\wedge_{x\notin Y_i}[(1 - R_\alpha(x, y)) \vee Y_i(x)]\}}\\ &= \sum_\alpha \frac{\wedge_{x\in X} R^*_\alpha(x, y)}{\wedge_{x\notin Y_i}(1 - R_\alpha(x, y))}.\end{aligned}$$

Hence,

$$\underline{\tilde{R}_3}(Y_1) = \frac{1/0.4}{e_1} + \frac{0.5/0}{e_2} + \frac{0.5/0}{e_3} + \frac{1/0.4}{e_4} + \frac{0.5/0.2}{e_5} + \frac{0.5/0}{e_6},$$
$$\underline{\tilde{R}_3}(Y_2) = \frac{1/0}{e_1} + \frac{0.5/0.2}{e_2} + \frac{0.5/0.2}{e_3} + \frac{1/0}{e_4} + \frac{0.5/0}{e_5} + \frac{0.5/0.2}{e_6},$$

and then

$$\underline{\tilde{R}_3}(Y_1) \cup \underline{\tilde{R}_3}(Y_2) = \frac{1/0.4}{e_1} + \frac{0.5/0.2}{e_2} + \frac{0.5/0.2}{e_3} + \frac{1/0.4}{e_4} + \frac{0.5/0.2}{e_5} + \frac{0.5/0.2}{e_6}.$$

Hence,

$$\gamma_3(Q) = \frac{|\underline{\tilde{R}_3}(Y_1) \cup \underline{\tilde{R}_3}(Y_2)|}{|X|} = \frac{\sum_{i=1}^{6} C_{\underline{\tilde{R}_3}(Y_1)\cup\underline{\tilde{R}_3}(Y_2)}(e_i)}{|X|} = 1.2/6,$$

where $C_{\underline{\tilde{R}_3}(Y_1)\cup\underline{\tilde{R}_3}(Y_2)}(e_i)$ is the cardinality of $\underline{\tilde{R}_3}(Y_1) \cup \underline{\tilde{R}_3}(Y_2)](e_i)$, which is defined as the product of the primary membership and its associated secondary grade.

Note that attribute $\tilde{R}_3$ causes the greatest increase in type-2 fuzzy rough dependency degree. Hence, this attribute is chosen and added to the potential reduction set. The process iterates, yielding $\gamma_{13} = 2.85/6$, $\gamma_{23} = 2.45/6$.

Because a larger increase of type-2 fuzzy rough dependency degree is produced by adding R_1 to the reduction candidate set, the new reduction candidate set becomes $\{R_1, \tilde{R}_3\}$.

Finally, adding $\tilde{R}_2$ to the potential reduction set gives $\gamma_{123}(Q) = 4.3/6$.

For the corresponding case of traditional rough attribute reduction depicted in Table 1, both $\{a_1, a_3\}$ and $\{a_2, a_3\}$ are reductions, $\{a_3\}$ is the core, and $\gamma'_{\{a_1,a_3\}}(Q) = \gamma'_{\{a_2,a_3\}}(Q) = 1$. Compared with the corresponding rough attribute reduction, type-2 fuzzy rough attribute reduction generates different dependency degrees for $\{a_1, a_3\}$ and $\{a_2, a_3\}$, but still affirms the importance of $\{a_3\}$.

5. Conclusions

In this paper, a general definition of type-2 fuzzy rough sets has been developed based on a wavy-slice representation (the Representation Theorem). Based on an arbitrary general type-2 fuzzy relation, a pair of upper and lower type-2 fuzzy rough approximation operators has also been derived. The granular structure of type-2 fuzzy rough sets has also been discussed, and it has been proven that the approximation operators in the type-2 fuzzy rough set model developed in this paper can be defined using the proposed basic information granules. Finally, attribute reduction within a type-2 fuzzy rough framework is discussed using a simple example.

So far, type-2 fuzzy sets can be represented by a vertical-slice representation, a wavy-slice representation, or an α-plane representations. Zhao and Xiao proposed a definition of type-2 fuzzy rough sets based on the α-plane representation method [26], and Wang investigated type-2 fuzzy rough sets on two finite universes of discourse based on extended t-norms with respect to type-2 fuzzy relations with convex normal fuzzy truth values using a vertical-slice representation of type-2 fuzzy sets [19]. These models simplify the computation of type-2 fuzzy rough sets but do not mention the granular structure of these sets because the concept of a type-2 fuzzy point is not clear in the α-plane or in the vertical-slice representations of type-2 fuzzy sets. This paper presents a new model for type-2 fuzzy rough sets based on wavy slices (the Representation Theorem). The proposed model decomposes type-2 fuzzy sets into ensembles of embedded type-2 sets, and the definition of fuzzy points is easy to extend to a type-2 fuzzy framework. Hence, the granular structure of type-2 fuzzy rough sets is creatively discussed, and it is shown that the upper and lower approximation operators can be defined as unions and intersections of certain granules. The proposed model illuminates the innate granular structure of type-2 fuzzy rough sets and proposes a set of basic granules in the framework of type-2 fuzzy rough sets.

As mentioned by Mendel in [11], the Representation Theorem is very useful for deriving theoretical results, but is not recommended for computational purposes because it would require explicit enumeration of the n_A embedded type-2 sets and n_A can be astronomical. For the same reason, the definition of type-2 fuzzy rough sets presented here is useful only for deriving theoretical results, the examples presented in this paper are simple, and the stated conclusions are all based on discrete type-2 fuzzy sets. Corresponding results for general type-2 fuzzy sets will be discussed in future work by the authors. Future work by the authors will consider the structure of attribute reduction in terms of granular type-2 fuzzy sets and various approaches to knowledge discovery in complex fuzzy information systems.

Acknowledgements

The authors would like to thank Professor Witold Pedrycz and the anonymous referees for their valuable comments and suggestions which have significantly improved the quality and presentation of this paper. The works described in this paper are supported by the National Natural Science Foundation of China (Nos. 61272095 and 61175067), Shanxi Scholarship Council of China (2013-014) and Shanxi Science and Technology Infrastructure (2015091001-0102).

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