Decision implication canonical basis: a logical perspective

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A R T I C L E I N F O	A B S T R A C T
	Due to its special role on logical deduction and practical applications of attribute implications, canonical basis has attracted much attention and been widely studied in Formal Concept Analysis. Canonical basis is constructed on pseudo-intents and, as an attribute implication basis, possesses of many important features, such as completeness, non-redundancy and minimality among all complete sets of attribute implications. In this paper, to deduce an analogous basis for decision implications, we introduce the notion of decision premise and form the so-called *decision implication canonical basis*. Furthermore, we show that the basis is complete, non-redundant and minimal among all complete sets of decision implications. We also present an algorithm to generate this canonical basis and analyze time complexity of this algorithm.

1. Introduction

R. Wille [26] introduced Formal Concept Analysis (FCA) as an order-theoretic method for the mathematical analysis of binary data. The starting point of FCA is based on a formalization of the philosophical understanding of a *concept* as a unit of thought constituted by its *extent* and *intent*. The extent of a concept is understood as the set of all objects belonging to the concept and the intent as the multitude of all attributes common to all those objects. The transformation from two-dimensional incidence tables to *concept lattices* structure is a crucial keystone from which FCA derives much of its power and versatility as a modeling tool. The concept lattices obtained the way turn out to be exactly the complete lattices, and the particular way in which they structure and represent knowledge is very appealing and natural from the perspective of many scientific disciplines. Over the past thirty years, FCA has been widely studied [9,8,20,24,4] and become a powerful tool for machine learning [11,29], software engineering [24] and information retrieval [4].

One of the aspects of FCA is attribute logic on attribute implication [22,9]. In FCA, an attribute implication is of the form $A \to B$, meaning that one can derive B from A. Then an attribute implication is *valid* in a set of data, if no data violates the attribute implication. A set of valid attribute implications may also be *complete* and *non-redundant*. A complete set of attribute implications carries all information from the data and thus one can reconstruct the data from the set, whereas a non-redundant set is a compact representation of attribute implications, indicating that the attribute implications in the set are independent of one another.

Concerning complete sets of attribute implications, [19] provided a complete, but redundant set of attribute implications, and described an algorithm for generating the set with a NextClosure-based algorithm [7]. More well-known is a complete and non-redundant set generated from *proper premises* [9], which produces its premises by removing some redundant information from closures. Pseudo-intent, introduced by V. Duquenne and J.L. Guigues [6], has obtained wide interests [9,12,2],

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Table 1
A formal context.

	a_1	a_2	a_3	a_4	a_5	a_6	d_1	d_2
x_1	×	×			×	×	×	
x_2			×					×
x_3	×		×			×	×	
x_4		×						
x_5		×	×	×	×	×	×	×
x_6			×		×			
x_7		×	×	×		×		×
x_8	×	×		×	×	×	×	

because it corresponds to an optimal representation of implications. Canonical basis, whose premises are exactly pseudo-intents, is proven to be complete, non-redundant, and more importantly, minimal among all complete sets of implications. However, finding a pseudo-intent is not an easy problem. There are some open problems [18] concerning the complexity of generating and finding pseudo-intents. For example, it has been proven that checking whether a subset is a pseudo-intent is coNP-complete [15,2], and that counting the number of pseudo-intents is even #P-hard [12,14]. To overcome this problem, Obiedkov etc. [16] showed some "genealogic" properties of attribute implications and presented an attribute-incremental algorithm for computing canonical basis. Experiment results showed that this algorithm is quite competitive. Valtchev etc. [25] adapted the divide-and-conquer policy and presented a method for computing canonical basis, which outperformed NextClosure on some datasets.

On the other hand, decision-based FCA (including decision context and decision implication) has been widely studied [10,11,13,20,28,27]. In the literature [20], Qu etc. presented a special inference rule, called α-decision inference rule, which may deduce other decision implications by enlarging premises of implications and/or reducing corresponding consequences. In the setting, [20] obtained an α-complete and α-non-redundant set of decision implications and showed that the complete set can be characterized by minimal generators [21,23,5]. In addition, an algorithm for generating the complete set was then given based on minimal generators and NextClosure algorithm. Afterwards, Zhai etc. [27] formulated decision implications and presented logical characteristics of decision implications. Specifically, [27] introduced the notions of *closure* and *unite closure* and established the semantical aspect of decision implications; [27] then formed two deduction rules and showed that the two rules are complete with respect to the semantical aspect, which established the syntactical aspect of decision implications.

Following [20] and [27], the paper intends to construct a canonical basis for decision implications. This canonical basis, called decision implication canonical basis is with the so-called *decision premises* (*d-premises*) as its premises of decision implications and closures on decision subcontext as its consequences of decision implications. Decision implication canonical basis is semantically complete and non-redundant, and furthermore it contains the least number of decision implications among all complete subsets of implications. In other words, d-premise is a counterpart of pseudo-intent in the case of decision implications.

This paper is organized as follows. Section 2 presents some basic notions about FCA and decision contexts, which are taken from [9,13,20]. We reformulate decision implication in terms of logic in Section 3. In Section 4, we introduce the notion of decision premise and prove that the so-called decision implication canonical basis is complete and non-redundant. Besides, we also show that the canonical basis contains the least number of decision implications among all complete subsets of implications. Section 5 discusses how to generate d-premises and decision implication canonical basis. Section 6 concludes the paper and lists some further remarks.

2. Formal concept analysis

2.1. Basic notions of FCA

This subsection provides a brief overview of FCA, and for more extensive introduction refer to [9].

A triple $K = (G, M, I)$ is a formal context, if G and M are sets, and $I \subseteq G \times M$ is a binary relation. In the case, the elements of G are called objects, the elements of M are called attributes, and I is viewed as an incidence relation between objects and attributes.

Example 1. Formal contexts are mostly represented by rectangular tables and an example is illustrated by Table 1. In the table, a cross means that the row object has the column attribute.

Within formal context, we can define some operators on object subsets and attribute subsets. Specifically, for a subset $A \subseteq G$ of objects we define:

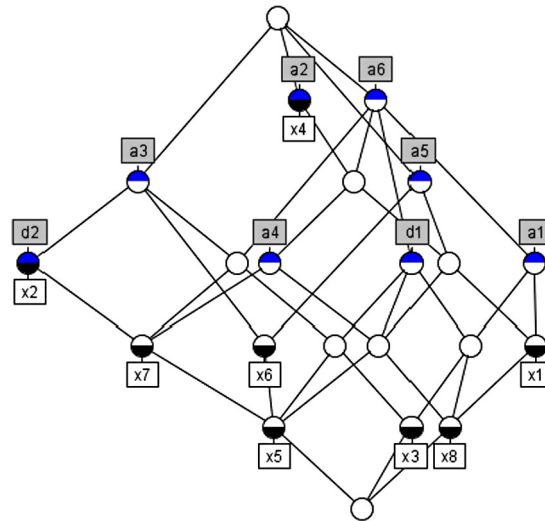$$A^I = \{m \in M \mid gIm, \forall g \in A\}$$

Fig. 1. Hasse diagram of concept lattice of context in Example 1.

that is, the set of attributes common to the objects in A. Correspondingly, for a subset $B \subseteq M$ we define:

$$B^I = \{g \in G \mid gIm, \forall m \in B\}$$

that is, the set of objects that have all attributes in B.

A pair $\mathfrak{C} = (A, B)$ is called a formal concept of K, if $A^I = B, B^I = A$. In the case, A is the intent of $\mathfrak{C}$ and B the extent of $\mathfrak{C}$. $\mathfrak{B}(K)$ denotes the set of all concepts of K.

Formal concepts can be partially ordered in a natural way. For two concepts $\mathfrak{C}_1 = (A_1, B_1), \mathfrak{C}_2 = (A_2, B_2) \in \mathfrak{B}(K)$, we define:

$$\mathfrak{C}_1 \preceq \mathfrak{C}_2 \iff A_1 \subseteq A_2 \ (\iff B_1 \supseteq B_2).$$

In the case, $\mathfrak{C}_2$ is a superconcept of $\mathfrak{C}_1$ and $\mathfrak{C}_1$ is a subconcept of $\mathfrak{C}_2$. The relation '$\preceq$' is called the hierarchical order of the concepts. The set of all concepts ordered in the way is called the concept lattice of K.

Example 2. Fig. 1 illustrates Hasse diagram representation of concept lattice of Table 1.

An attribute implication between attributes in M is a pair of subsets of M, denoted by $B_1 \to B_2$. The set B_1 is the premise of the implication $B_1 \to B_2$, and B_2 is its conclusion. Formally,

Definition 1. Let $K = (G, M, I)$ be a formal context, $B_1, B_2 \subseteq M$. $B_1 \to B_2$ is called an attribute implication of K if each object having all attributes from B_1 also has all attributes from B_2.

We can easily check the following theorem, and in the sequel we may make use of them without quotation.

Theorem 1. *Let $K = (G, M, I)$ be a formal context, A, A_1, A_2 are sets of objects and B, B_1, B_2 are set of attributes. Then:*

$$(1) \quad A_1 \subseteq A_2 \Rightarrow A_2^I \subseteq A_1^I \qquad (1') \quad B_1 \subseteq B_2 \Rightarrow B_2^I \subseteq B_1^I$$

$$(2) \quad A \subseteq A^{II} \qquad (2') \quad B \subseteq B^{II}$$

$$(3) \quad A^I = A^{III} \qquad (3') \quad B^I = B^{III}$$

$$(4) \quad A \subseteq B^I \iff B \subseteq A^I$$

2.2. Decision contexts and decision implications

In this subsection, we provide some notions such as decision context and decision implication [10,11,13,20,27].

Definition 2. (See [20].) A formal context $K = (G, M, I)$ is called a decision context if $M = C \cup D$, $C \cap D = \emptyset$ and $I = I_C \cup I_D$, where C is the set of condition attributes, D is the set of decision attributes, $I_C \subseteq G \times C$ is the set of condition incidence relations, and $I_D \subseteq G \times D$ is the set of decision incidence relations.
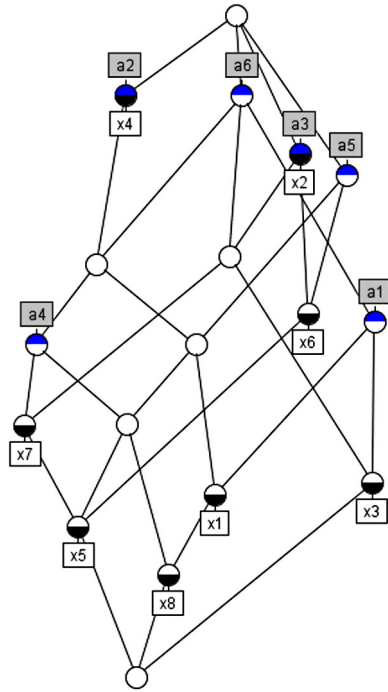
Fig. 2. Hasse diagram of concept lattice of subcontext K_C in Example 1.
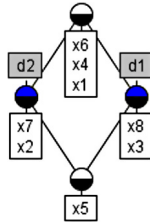

Fig. 3. Hasse diagram of concept lattice of subcontext K_D in Example 1.

Clearly, a decision context consists of two sub-contexts, the condition sub-context $K_C = (G, C, I_C)$ and the decision sub-context $K_D = (G, D, I_D)$. For $A \subseteq G$, $B_{C1} \subseteq C$ and $B_{D1} \subseteq D$, the symbols $A^{I_C}, A^{I_D}, B_{C1}^{I_C}, B_{D1}^{I_D}$ will be abbreviated to $A^C, A^D, B_{C1}^C, B_{D1}^D$.

Example 3. Take Table 1 as a decision context, in which the condition attributes are $\{a1, a2, \cdots, a6\}$ and the decision attributes are $\{d1, d2\}$. Hasse diagrams of concept lattices of subcontexts are shown in Fig. 2 and Fig. 3 respectively.

Definition 3. A decision context $K = (G, C \cup D, I_C \cup I_D)$ is consistent if for all $g, h \in G$, $g^{CC} = h^{CC}$ implies $g^{DD} = h^{DD}$.

Definition 3 expresses that, in a consistent decision context, if two objects possess the same condition attributes (i.e. $g^{CC} = h^{CC}$), then their decision attributes are also identical. That is to say, when we make some decisions (i.e., decision implication, see below), the same conditions (i.e., premises) will result in the same decisions (i.e., consequences). Throughout our paper, we assume that all decision contexts are consistent.

Also note that Definition 3 presents a similar notion of the absence of *hopeless* (positive or negative) examples (see [8] for details).

In consistent decision contexts, we can introduce decision implication.

Definition 4. Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, and B_1, B_2 are subsets of attributes. An implication $B_1 \rightarrow B_2$ of K is called a decision implication of K, if $B_1 \subseteq C$ and $B_2 \subseteq D$.

Example 4. The system of decision implications of Table 1 is shown in the following:

$$\emptyset \rightarrow \emptyset \qquad\qquad \{a_1, a_4\} \rightarrow \{d_1\} \qquad \{a_5, a_6\} \rightarrow \{d_1\}$$
$$\{a_1\} \rightarrow \{d_1\} \qquad \{a_1, a_5\} \rightarrow \{d_1\} \qquad \{a_1, a_2, a_3\} \rightarrow \{d_1, d_2\}$$
$$\{a_2\} \rightarrow \emptyset \qquad\; \{a_2, a_3\} \rightarrow \{d_2\} \qquad \{a_1, a_3, a_4\} \rightarrow \{d_1, d_2\}$$
$$\{a_3\} \rightarrow \emptyset \qquad\; \{a_2, a_4\} \rightarrow \{d_1\} \qquad \{a_1, a_3, a_5\} \rightarrow \{d_1, d_2\}$$
$$\{a_4\} \rightarrow \emptyset \qquad\; \{a_1, a_2\} \rightarrow \{d_1\} \qquad \{a_2, a_3, a_5\} \rightarrow \{d_1, d_2\}$$
$$\{a_5\} \rightarrow \emptyset \qquad\; \{a_3, a_4\} \rightarrow \{d_2\} \qquad \{a_3, a_4, a_5\} \rightarrow \{d_1, d_2\}$$
$$\{a_6\} \rightarrow \emptyset \qquad\; \{a_1, a_3\} \rightarrow \{d_1\} \qquad \{a_3, a_5, a_6\} \rightarrow \{d_1, d_2\}$$
$$\{a_2, a_6\} \rightarrow \emptyset \qquad \{a_3, a_6\} \rightarrow \emptyset$$
$$\{a_3, a_5\} \rightarrow \emptyset \qquad \{a_4, a_5\} \rightarrow \{d_1\}$$

We can easily prove the following characteristic of decision implications.

Proposition 1. *Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, $B_1 \subseteq C$ and $B_2 \subseteq D$. Then $B_1 \rightarrow B_2$ is a decision implication of K if and only if $B_1^C \subseteq B_2^D$, if and only if $B_2 \subseteq B_1^{CD}$.*

3. Semantic aspects of decision implications

For developing canonical bases for decision implications, we first need to formulate decision implication in terms of logic.

Definition 5. (See [9,27].) *For an attribute set M with $M = C \cup D$ and $C \cap D = \emptyset$, a decision implication between subsets of C and D is of the form $B_1 \rightarrow B_2$ such that $B_1 \subseteq C$ and $B_2 \subseteq D$. Here B_1 is the premise of the decision implication and B_2 the consequence of the decision implication. A subset $T \subseteq M$ respects $B_1 \rightarrow B_2$, denoted by $T \models B_1 \rightarrow B_2$, if $B_1 \nsubseteq T \cap C$ or $B_2 \subseteq T \cap D$ (equivalently, $B_1 \subseteq T \cap C$ implies $B_2 \subseteq T \cap D$). T respects a set $\mathcal{L}$ of decision implications, denoted by $T \models \mathcal{L}$, if $T \models B_1 \rightarrow B_2$, for any $B_1 \rightarrow B_2 \in \mathcal{L}$. $B_1 \rightarrow B_2$ holds in a set $\{T_1, T_2, \cdots, T_n\}$ if each T_i respects $B_1 \rightarrow B_2$.*

Different from Definition 36 in [9], decision implication is constructed on, instead of M, two disjoint subsets of M. Think of M as a set of attributes; C and D are then called condition attributes and decision attributes respectively in accordance with Definition 2.

Note that a decision implication of the form $B_1 \rightarrow B_2$ is merely a syntactical formula without any meaning. The validity of decision implication has its effect only after defining the notion "respect". Then we say, a decision implication $B_1 \rightarrow B_2$ is *valid* with respect to a subset T if and only if $T \models B_1 \rightarrow B_2$. In this case, we also say that T is an *interpretation* or a *model* of $B_1 \rightarrow B_2$.

Note that the above definition is unrelated to decision context. Thus, applying the above notion to decision context, we say:

Definition 6. *A decision implication holds in a decision context if it holds in the system $\{g^C \cup g^D \mid g \in G\}$.*

Intuitively, a decision implication holds in a decision context if all objects having attributes from the premise also have the attributes from the consequence.

Although Definition 6 seems different from Definition 4, we can conclude that

Theorem 2. *(See [9].) For a decision context K and $B_1 \subseteq C$, $B_2 \subseteq D$, the following statements are equivalent:*

1. *The system $\{A^C \cup A^D \mid A \subseteq G\}$ respects $B_1 \rightarrow B_2$ (seems stricter than Definition 6);*
2. *$B_1 \rightarrow B_2$ holds in K (Definition 6);*
3. *$B_1 \rightarrow B_2$ is a decision implication of K (Definition 4).*

Generally speaking, the number of decision implications in a decision context is quite large, since there are lots of *redundant* decision implications, which can be deduced from other decision implications.

Definition 7. (See [27].) *A decision implication $B_1 \rightarrow B_2$ semantically follows from a set $\mathcal{L}$ of decision implications if for any $T \subseteq M$, $T \models \mathcal{L}$ implies $T \models B_1 \rightarrow B_2$, denoted by $\mathcal{L} \vdash B_1 \rightarrow B_2$. A set $\mathcal{L}$ of decision implications is closed if any decision implication following from $\mathcal{L}$ is contained in $\mathcal{L}$. $\mathcal{L}$ is non-redundant if any $B_1 \rightarrow B_2 \in \mathcal{L}$ does not follow from $\mathcal{L} \setminus \{B_1 \rightarrow B_2\}$. For a closed set $\mathcal{L}$ of decision implications, a subset $\mathcal{D} \subseteq \mathcal{L}$ is complete with respect to $\mathcal{L}$ if $\mathcal{L}$ semantically follows from $\mathcal{D}$.*

Transferring the definition to the case of decision context, we have the counterpart of "closed".

Definition 8. A set $\mathcal{L}$ of decision implications of K is complete if any decision implication of K follows from $\mathcal{L}$. A set $\mathcal{L}$ of decision implications of K is non-redundant if any $B_1 \to B_2 \in \mathcal{L}$ does not follow from $\mathcal{L} \backslash \{B_1 \to B_2\}$.

In classic FCA [9], a set $\mathcal{L}$ of implications is complete if and only if every set respecting $\mathcal{L}$ is an intent. In the case of decision context, this is not true, since, for example, all subsets of D will respect all decision implications of K.

As an application of Definition 7, we can prove the following theorem.

Theorem 3. *Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context, $B_{C1}, B_{C2} \subseteq C$, $B_{D1}, B_{D2} \subseteq D$, and $B_{C1} \to B_{D1}$ is a decision implication of K. If $B_{C1} \subseteq B_{C2}$ and $B_{D1} \supseteq B_{D2}$, then $B_{C2} \to B_{D2}$ is also a decision implication of K.*

Proof. Let $T \models B_{C1} \to B_{D1}$. If $B_{C1} \nsubseteq T$, then $B_{C2} \nsubseteq T$, since $B_{C2} \supseteq B_{C1}$. Otherwise, if $B_{D1} \subseteq T$, by $B_{D1} \supseteq B_{D2}$, $B_{D2} \subseteq T$. $\square$

In fact, Theorem 3 was first shown in [20] with the name "α-decision inference rule", but as an inference rule, instead of a semantical counterpart.

4. D-premise and decision implication canonical basis

In this section, we introduce the notion of decision premise and show that a set of decision implications, called *decision implication canonical basis* consisting of all decision implications whose premises are decision premises and whose consequences are the closures of the decision premises in the decision subcontext, are complete and non-redundant. Furthermore, we will show an interesting characteristic of this basis, namely the set contains the least number of decision implications among all complete sets of decision implications.

We first present the notion of decision premise in terms of logic, and then apply it to decision context.

Definition 9. (See [27].) For a set $\mathcal{L}$ of decision implications, the closure of $P \subseteq C$ with respect to $\mathcal{L}$ is the set

$$P^{\mathcal{L}} = \bigcup \{B_2 \mid B_1 \to B_2 \in \mathcal{L} \text{ and } B_1 \subseteq P\}.$$

And the set $P \cup P^{\mathcal{L}}$ is called the unite closure of P with respect to $\mathcal{L}$.

Definition 10. Let $\mathcal{L}$ be a set of decision implications on C and D. An $\mathcal{L}$-decision premise of $\mathcal{L}$ is a subset $P \subseteq C$ such that

1. P is minimal with respect to $P^{\mathcal{L}}$, i.e., if $Q \subset P$, then $Q^{\mathcal{L}} \subset P^{\mathcal{L}}$;
2. P is proper, i.e.,

$$P^{\mathcal{L}} \neq \bigcup \{B_1^{\mathcal{L}} \mid B_1 \text{ is an } \mathcal{L}\text{-decision premise and } B_1 \subset P\}.$$

To apply Definition 10 to decision context, we need the following lemmas.

Lemma 1. *Let K be a decision context. Then $B_1 \to B_2$ is a decision implication of K if and only if for any $P \subseteq C$, $P \cup P^{CD} \models B_1 \to B_2$.*

Proof. Necessity: To prove $P \cup P^{CD} \models B_1 \to B_2$, it suffices to show that $B_1 \subseteq P$ implies $B_2 \subseteq P^{CD}$. Since $B_1 \subseteq P$, we have $B_1^{CD} \subseteq P^{CD}$. Since $B_1 \to B_2$ is a decision implication of K, we obtain $B_2 \subseteq B_1^{CD}$ by Proposition 1, which yields $B_2 \subseteq P^{CD}$.

Sufficiency: Since for any $P \subseteq C$, we have $P \cup P^{CD} \models B_1 \to B_2$. Setting $P = B_1$, we obtain $B_1 \cup B_1^{CD} \models B_1 \to B_2$, yielding $B_2 \subseteq B_1^{CD}$. By Proposition 1, $B_1 \to B_2$ is a decision implication of K. $\square$

Lemma 2. *(See [27].) Let $\mathcal{L}$ be a set of decision implications. Then a subset $T \subseteq C \cup D$ respects $\mathcal{L}$ if and only if $(T \cap C)^{\mathcal{L}} \subseteq T \cap D$. In particular, for any $P \subseteq C$, $P \cup P^{\mathcal{L}} \models \mathcal{L}$.*

Lemma 3. *(See [27].) Let $\mathcal{L}$ be a set of decision implications and $P \subseteq C$. Then $\mathcal{L} \vdash P \to P^{\mathcal{L}}$.*

Lemma 4. *Let K be a decision context and $\mathcal{L}$ be a complete set of decision implications of K. Then for any $P \subseteq C$, we have $P^{CD} = P^{\mathcal{L}}$.*

Proof. By Proposition 1, we know that $P \to P^{CD}$ is a decision implication of K. By definition of completeness, $P \to P^{CD}$ follows from $\mathcal{L}$. Since $P \cup P^{\mathcal{L}}$ is a model of $\mathcal{L}$ (by Lemma 2), $P \cup P^{\mathcal{L}}$ is also a model of $P \to P^{CD}$, which implies $P^{CD} \subseteq P^{\mathcal{L}}$.

Conversely, since each decision implication of $\mathcal{L}$ holds in K, we have $P \cup P^{CD} \models \mathcal{L}$ by Lemma 1. Since $\mathcal{L} \vdash P \to P^{\mathcal{L}}$ (by Lemma 3), we know $P \cup P^{CD} \models P \to P^{\mathcal{L}}$, i.e., $P^{\mathcal{L}} \subseteq P^{CD}$. $\square$

Now, since for any $P \subseteq C$, we have $P^{CD} = P^{\mathcal{L}}$, Definition 10 can be applied to decision context.

Definition 11. Let K be a decision context. A decision premise of K (d-premise for short) is a subset $P \subseteq C$ such that

1. P is minimal with respect to P^{CD}, i.e., if $Q \subset P$, then $Q^{CD} \subset P^{CD}$;
2. P is proper, i.e.,

$$P^{CD} \neq \bigcup \{ B_1^{CD} \mid B_1 \text{ is a d-premise and } B_1 \subset P \}.$$

Note that the two conditions above are independent just as the following example shows.

Example 5. Here we list all d-premises of Table 1 in the following:

$\emptyset$	$\{a_2, a_3\}$	$\{a_4, a_5\}$	$\{a_1, a_3, a_4\}$	$\{a_1, a_3, a_5\}$	$\{a_3, a_5, a_6\}$
$\{a_1\}$	$\{a_2, a_4\}$	$\{a_5, a_6\}$	$\{a_3, a_4, a_5\}$	$\{a_2, a_3, a_5\}$	

After simple checking, $\{a_1, a_2, a_3\}$ is minimal with respect to $\{a_1, a_2, a_3\}^{CD} = \{d_1, d_2\}$, but not proper, since we can merge the two consequences of decision implications, $\{a_1\} \to \{d_1\}$ and $\{a_2, a_3\} \to \{d_2\}$, and obtain $\{d_1, d_2\}$. On the other hand, there are lots of decision implications meeting condition 2 but violating condition 1, for example the subset $\{a_1, a_2\}$.

From each subset $P \subseteq C$, in particular for a d-premise, one can produce a *valid* decision implication $P \to P^{CD}$. Furthermore, the set of decision implications whose premise are d-premises is complete and non-redundant.

Definition 12. Let $\mathcal{L}$ be a set of decision implications on C and D. The set of decision implications

$$\mathcal{D}_{\mathcal{L}} = \{ P \to P^{\mathcal{L}} \mid P \text{ is a d-decision premise} \}$$

is called decision implication canonical basis on $\mathcal{L}$.

Applying Definition 12 to decision context, we have the following definition.

Definition 13. Let K be a decision context. The set of decision implications

$$\mathcal{D} = \{ P \to P^{CD} \mid P \text{ is a d-premise} \}$$

is called decision implication canonical basis of K.

Now we will prove that $\mathcal{D}_{\mathcal{L}}$ is complete and non-redundant. It is easy to see that the completeness and non-redundancy of $\mathcal{D}$ is just a consequence of this result. We first need some lemmas.

Definition 14. Let $\mathcal{L}$ be a set of decision implications on C and D. We define

$$\bar{\mathcal{L}} = \{ B_1 \to B_2 \mid \mathcal{L} \vdash B_1 \to B_2 \}$$

It is to check that $\bar{\mathcal{L}}$ is closed in the sense of Definition 7, and thus, we call $\bar{\mathcal{L}}$ the corresponding closed set of $\mathcal{L}$.

Lemma 5. *(See [27].) Let $\mathcal{L}$ be a set of decision implications. Then a decision implication $B_1 \to B_2$ follows from $\mathcal{L}$ if and only if $B_2 \subseteq B_1^{\mathcal{L}}$.*

Lemma 6. *Let $\mathcal{L}$ be a set of decision implications on C and D. If $T \models B_1 \to B_1^{\mathcal{L}}$ and $B_1 \to B_2 \in \bar{\mathcal{L}}$, then $T \models B_1 \to B_2$.*

Proof. If $B_1 \to B_2 \in \bar{\mathcal{L}}$, then $\mathcal{L} \vdash B_1 \to B_2$. Then by Lemma 5, we have $B_2 \subseteq B_1^{\mathcal{L}}$. To prove $T \models B_1 \to B_2$, we assume $B_1 \subseteq T \cap C$. Since $T \models B_1 \to B_1^{\mathcal{L}}$, we have $B_1^{\mathcal{L}} \subseteq T \cap D$, yielding $B_2 \subseteq B_1^{\mathcal{L}} \subseteq T \cap D$. $\square$

Now we prove the following theorem.

Theorem 4. *$\mathcal{D}_{\mathcal{L}}$ is complete and non-redundant with respect to $\bar{\mathcal{L}}$, where $\bar{\mathcal{L}}$ is the corresponding closed set of $\mathcal{L}$.*

Proof. Completeness: To prove the completeness of $\mathcal{D}_{\mathcal{L}}$, we need to prove that $B_1 \to B_2 \in \bar{\mathcal{L}}$ follows from $\mathcal{D}_{\mathcal{L}}$, i.e., $T \models \mathcal{D}_{\mathcal{L}}$ implies $T \models B_1 \to B_2$. By Lemma 6, it suffices to show $T \models B_1 \to B_1^{\mathcal{L}}$. We list three cases:

1. B_1 is minimal with respect to $B_1^{\mathcal{L}}$ and is proper. In this case, B_1 is an $\mathcal{L}$-decision premise and $B_1 \to B_1^{\mathcal{L}} \in \mathcal{D}_{\mathcal{L}}$. The assertion follows;

2. B_1 is minimal with respect to $B_1^{\mathcal{L}}$ but is not proper. Since B_1 is not proper, then we know

$$B_1^{\mathcal{L}} = \bigcup \{B_3^{\mathcal{L}} \mid B_3 \text{ is an } \mathcal{L}\text{-decision premise and } B_3 \subset B_1\}.$$

To prove $T \models B_1 \rightarrow B_2$, by Lemma 6, we need to prove $T \models B_1 \rightarrow B_1^{\mathcal{L}}$. Let us assume $B_1 \subseteq T$. Since $T \models \mathcal{D}_{\mathcal{L}}$, in particular, $T \models B_3 \rightarrow B_3^{\mathcal{L}}$ for each $\mathcal{L}$-decision premise B_3 with $B_3 \subset B_1$, then by $B_3 \subset B_1 \subseteq T$, we have $B_3^{\mathcal{L}} \subseteq T \cap D \subset T$. Thus $B_1^{\mathcal{L}} = \bigcup B_3^{\mathcal{L}} \subseteq T$, yielding $T \models B_1 \rightarrow B_1^{\mathcal{L}}$.
3. B_1 is not minimal with respect to $B_1^{\mathcal{L}}$. In the setting, we can find a minimal subset $B_3 \subset B_1$ with $B_3^{\mathcal{L}} = B_1^{\mathcal{L}}$. Next, replacing B_1 with B_3 and processing according to the case (2), one can confirm the result.

Non-redundancy: It remains to show that $\mathcal{D}_{\mathcal{L}}$ is non-redundant, i.e., $\mathcal{D}_{\mathcal{L}} \backslash \{B_1 \rightarrow B_1^{\mathcal{L}}\} \nvdash B_1 \rightarrow B_1^{\mathcal{L}}$, for $B_1 \rightarrow B_1^{\mathcal{L}} \in \mathcal{D}_{\mathcal{L}}$. Now we want to seek a model respecting $\mathcal{D}_{\mathcal{L}} \backslash \{B_1 \rightarrow B_1^{\mathcal{L}}\}$ but not respecting $B_1 \rightarrow B_1^{\mathcal{L}}$.
Set

$$T = B_1 \cup \bigcup \{B_3^{\mathcal{L}} \mid B_3 \text{ is an } \mathcal{L}\text{-decision premise and } B_3 \subset B_1\},$$

and then we assert that

1. $T \models \mathcal{D}_{\mathcal{L}} \backslash \{B_1 \rightarrow B_1^{\mathcal{L}}\}$, but
2. $T \nvDash B_1 \rightarrow B_1^{\mathcal{L}}$,

which actually means $\mathcal{D}_{\mathcal{L}} \backslash \{B_1 \rightarrow B_1^{\mathcal{L}}\} \nvdash B_1 \rightarrow B_1^{\mathcal{L}}$.

For the first assertion, we take a decision implication $B_3 \rightarrow B_3^{\mathcal{L}} \in \mathcal{D}_{\mathcal{L}} \backslash \{B_1 \rightarrow B_1^{\mathcal{L}}\}$ and need to prove $T \models B_3 \rightarrow B_3^{\mathcal{L}}$. If $B_3 \subseteq T \cap C = B_1$, then $B_3 \subset B_1$ (obviously, $B_3 \neq B_1$) and thus $B_3^{\mathcal{L}} \subseteq \bigcup_{B_3 \subset B_1} B_3^{\mathcal{L}} = T \cap D$.

For the second assertion, since B_1 is an $\mathcal{L}$-decision premise, then we have

$$B_1^{\mathcal{L}} \supset \bigcup \{B_3^{\mathcal{L}} \mid B_3 \text{ is an } \mathcal{L}\text{-decision premise and } B_3 \subset B_1\} = V$$

There must exist an attribute $m \in B_1^{\mathcal{L}} \backslash V$. In this case, we have $B_1 \subseteq T \cap C = B_1$, and $m \in B_1^{\mathcal{L}} \nsubseteq T \cap D = V$, i.e., $T \nvDash B_1 \rightarrow B_1^{\mathcal{L}}$. □

Similarly, Theorem 4 holds when starting from a decision context and a complete set of decision implications:

Corollary 1. *Let K be a decision context and $\mathcal{L}$ a complete set of decision implications of K. Then decision implication canonical basis of K is complete and non-redundant.*

Example 6. The decision implication canonical basis for Table 1 is shown in the following:

$\emptyset \rightarrow \emptyset$	$\{a_4, a_5\} \rightarrow \{d_1\}$	$\{a_1, a_3, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_1\} \rightarrow \{d_1\}$	$\{a_5, a_6\} \rightarrow \{d_1\}$	$\{a_2, a_3, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_2, a_3\} \rightarrow \{d_2\}$	$\{a_1, a_3, a_4\} \rightarrow \{d_1, d_2\}$	$\{a_3, a_4, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_2, a_4\} \rightarrow \{d_1\}$	$\{a_3, a_5, a_6\} \rightarrow \{d_1, d_2\}$	

For comparison purpose, we also list the system of α-maximal decision implications [20]:

$\emptyset \rightarrow \emptyset$	$\{a_4, a_5\} \rightarrow \{d_1\}$	$\{a_1, a_3, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_1\} \rightarrow \{d_1\}$	$\{a_5, a_6\} \rightarrow \{d_1\}$	$\{a_2, a_3, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_2, a_3\} \rightarrow \{d_2\}$	$\{a_1, a_2, a_3\} \rightarrow \{d_1, d_2\}$	$\{a_3, a_4, a_5\} \rightarrow \{d_1, d_2\}$
$\{a_2, a_4\} \rightarrow \{d_1\}$	$\{a_1, a_3, a_4\} \rightarrow \{d_1, d_2\}$	$\{a_3, a_5, a_6\} \rightarrow \{d_1, d_2\}$

It is easily seen that $\{a_1, a_2, a_3\} \rightarrow \{d_1, d_2\}$ has been removed from the list of decision implication canonical basis, since it can be obtained by merging two decision implications, $\{a_1\} \rightarrow \{d_1\}$ and $\{a_2, a_3\} \rightarrow \{d_2\}$.

It turns out that d-premise is a counterpart of pseudo-intent in the setting of decision contexts. Theorem 4 and Corollary 1 have shown that $\mathcal{D}_{\mathcal{L}}$ and $\mathcal{D}$ are complete and non-redundant. The following theorem will show the optimal characteristic of $\mathcal{D}_{\mathcal{L}}$.

Theorem 5. *Let $\mathcal{L}$ be a set of decision implications on C and D, and P be an $\mathcal{L}$-decision premise of $\mathcal{L}$. Then we have $P \rightarrow P^{\mathcal{L}} \in \mathcal{L}$ or $\mathcal{L} \nvdash P \rightarrow P^{\mathcal{L}}$.*

Proof. We assume $P \to P^{\mathcal{L}} \notin \mathcal{L}$ and need to prove $\mathcal{L} \nvdash P \to P^{\mathcal{L}}$. It suffices to show that $T \models \mathcal{L}$ but $T \nvDash P \to P^{\mathcal{L}}$. Now we define

$$\mathcal{L}' = \left\{ B_1 \to B_1^{\mathcal{L}} \mid B_1 \to B_2 \in \mathcal{L} \right\}$$

By Lemma 6, it is easy to see that if $T \models \mathcal{L}'$, then $T \models \mathcal{L}$. We thus need to prove if $T \models \mathcal{L}'$, then $T \nvDash P \to P^{\mathcal{L}}$.

Let $P \to P^{\mathcal{L}} \notin \mathcal{L}$ and $T \models \mathcal{L}'$. Assume $P \subseteq T$. Since P is an $\mathcal{L}$-decision premise, then

$$P^{\mathcal{L}} \supset \bigcup \left\{ B_1^{\mathcal{L}} \mid B_1 \text{ is an } \mathcal{L}\text{-decision premise and } B_1 \subset P \right\}.$$

We now assert that

$$\begin{aligned}
\bigcup \left\{ B_1^{\mathcal{L}} \mid B_1 \text{ is an } \mathcal{L}\text{-decision premise and } B_1 \subset P \right\} &= W \\
\supseteq \bigcup \left\{ B_1^{\mathcal{L}} \mid B_1 \to B_1^{\mathcal{L}} \in \mathcal{L}' \right\} &= V
\end{aligned} \tag{1}$$

which will be proven below. If Eq. (1) holds, then $P^{\mathcal{L}} \supset V$ and there exists $m \in P^{\mathcal{L}} \backslash V$. Set

$$T = P \cup \bigcup \left\{ B_1^{\mathcal{L}} \mid B_1 \text{ is an } \mathcal{L}\text{-decision premise and } B_1 \subset P \right\},$$

and then, similar to the proof of Theorem 4, we can show that $T \models \mathcal{L}'$ but $T \nvDash P \to P^{\mathcal{L}}$, which yields $\mathcal{L}' \nvdash P \to P^{\mathcal{L}}$ and thus $\mathcal{L} \nvdash P \to P^{\mathcal{L}}$.

Now let us prove Eq. (1). For each $B_1 \to B_1^{\mathcal{L}} \in \mathcal{L}'$, three cases are possible:

1. B_1 is an $\mathcal{L}$-decision premise. In this case, $B_1^{\mathcal{L}} \subseteq W$ and the conclusion holds;
2. B_1 is minimal with respect to $B_1^{\mathcal{L}}$ but not proper. In this case, we have

   $$B_1^{\mathcal{L}} = \bigcup \left\{ B_3^{\mathcal{L}} \mid B_3 \text{ is an } \mathcal{L}\text{-decision premise and } B_3 \subset B_1 \right\},$$

   which means that $B_1^{\mathcal{L}}$ can be formed by union of some subsets of W, and thus Eq. (1) holds;
3. B_1 is not minimal. Then we can find a minimal subset $B_3 \subset B_1$ with $B_3^{\mathcal{L}} = B_1^{\mathcal{L}}$, and following, the process of case 2, we can complete the proof. □

Let $\mathcal{L}$ be a set of decision implications on C and D, and $\bar{\mathcal{L}}$ be the closed set of $\mathcal{L}$. Following the theorem above, we have:

Theorem 6. $\mathcal{D}_{\mathcal{L}}$ *contains the least number of decision implications among all complete sets of $\bar{\mathcal{L}}$.*

Proof. Let $\mathcal{F}$ be a complete set of $\bar{\mathcal{L}}$ and P be an $\mathcal{L}$-decision premise. By Theorem 3 of [27], we have $P^{\mathcal{L}} = P^{\mathcal{F}}$ for any $P \subseteq C$. Thus, by Lemma 3, we have $\mathcal{F} \vdash P \to P^{\mathcal{F}}(= P^{\mathcal{L}})$, and, by Theorem 5, we have $P \to P^{\mathcal{F}}(= P^{\mathcal{L}}) \in \mathcal{F}$. Thus $P \to P^{\mathcal{L}}$ has to be in $\mathcal{F}$ since $\mathcal{F}$ is complete. In view of arbitrariness of P, we know each decision implication from $\mathcal{D}_{\mathcal{L}}$ is contained in $\mathcal{L}$. Thus $\mathcal{D}_{\mathcal{L}}$ contains the least number of decision implications among all complete sets of $\bar{\mathcal{L}}$. □

Similarly, we have:

Corollary 2. *Let K be a decision context. Then $\mathcal{D}$ contains the least number of decision implications among all complete sets of K.*

Theorems 4 and 6 state that $\mathcal{D}_{\mathcal{L}}$ is a counterpart of pseudo-intent in terms of logic, whereas Corollaries 1 and 2 state that $\mathcal{D}$ is a counterpart of pseudo-intent in term of decision contexts. In other words, decision implication canonical basis is a natural basis for decision implications.

5. Generating decision implication canonical basis

To generate decision implication canonical basis, an intuitive way is to check validity of all subsets of C according to Definition 11. However the approach is quite impractical even for not so large-scale contexts.

An alternative way is based on [20], where we provided an algorithm to compute the system Σ of α-maximal decision implications.

First we adopt a sufficient and necessary condition for α-maximal decision implication, taken from [20].

Lemma 7. *Let $K = (G, C \cup D, I_C \cup I_D)$ be a decision context. Then $B_{C1} \to B_{D1}$ is an α-maximal decision implication of K if and only if*

1. B_{C1} *is a minimal generator of the condition sub-context* K_C;
2. $B_{D1} = B_{C1}^{CD}$;
3. *If* $B_{C2} \subset B_{C1}$, *then* $B_{C1}^{CD} \neq B_{C2}^{CD}$.

In fact, the condition (1) of Definition 11 is the same as the condition (3) of Lemma 7, and moreover it implies the condition (1) of Lemma 7:

Theorem 7. *If* B_{C1} *is minimal with respect to* B_{C1}^{CD}, *then* P *is a minimal generator of the condition sub-context* K_C

Proof. If B_{C1} is not a minimal generator of (B_{C1}^C, B_{C1}^{CC}), then there exists a minimal subset $B_{C2} \subset B_{C1}$ such that $B_{C2}^{CC} = B_{C1}^{CC}$, which yields $B_{C2}^C = B_{C2}^{CCC} = B_{C1}^{CCC} = B_{C1}^C$. Thus $B_{C2}^{CD} = B_{C1}^{CD}$, which contradicts with the fact that B_{C1} is minimal with respect to B_{C1}^{CD}. $\quad \square$

Thus decision implication canonical basis is contained in the system Σ of α-maximal decision implications, and in order to obtain decision implication canonical basis, the remaining work is to check Σ and remove the decision implications that violates condition (2) of Definition 11, as shown by Algorithm 1.

Algorithm 1 Generating Decision Implication Canonical Basis Based on Minimal Generators (GDCBonMG).

Input: Decision context K
Output: Decision implication canonical basis $\mathcal{D}$;

1: $\mathcal{D} = \emptyset$ {decision implication canonical basis}
2: $\mathcal{M} = \emptyset$ {the set of minimal generators}
3: $\mathcal{T} = \emptyset$ {accumulating variable for checking condition (2) of Definition 11}
4: Generate all minimal generators $\mathcal{M}$ of (G, C, I_C) [21,23,5] and sort $\mathcal{M}$ in lectical order
5: **for all** $B_{C1} \in \mathcal{M}$ **do**
6:     **for all** $B_{C2} \in \mathcal{D}$ with $B_{C2} \subset B_{C1}$ **do**
7:         **if** $B_{C1}^{CD} \neq B_{C2}^{CD}$ **then**
8:             $\mathcal{T} = T \cup \{B_{C2}^{CD}\}$
9:         **else**
10:            break
11:         **end if**
12:     **end for**
13:     Remove B_{C1} from $\mathcal{M}$
14:     **if** $B_{C1} \neq \mathcal{T}$ **then**
15:         Generate decision implication $B_{C1} \rightarrow B_{C1}^{CD}$ and add to $\mathcal{D}$
16:     **end if**
17: **end for**
18: **return** $\mathcal{D}$

In line 4 of Algorithm 1, the time complexity of extracting and sorting minimal generators is, at most,

$$O\left(|M| \cdot \left(db + \binom{|M|}{\lfloor \frac{|M|}{2} \rfloor} \cdot |G| \cdot |M| \right) \right)$$

where db is the access time of the formal context. After the optimization of sort in line 4, the time-consuming of processing lines 5–17, which clearly depends on the number of minimal generators, will be, at worst, $O(|\mathcal{M}|^2)$, where $\mathcal{M}$ is the number of minimal generators. To sum up, Algorithm 1 will have the time complexity of

$$O\left(|M| \cdot \left(db + \binom{|M|}{\lfloor \frac{|M|}{2} \rfloor} \cdot |G| \cdot |M| \right) + |\mathcal{M}|^2 \right)$$

So the algorithm is also impractical especially for large contexts.

6. Conclusion and further remarks

In this paper, we introduce the notion of "d-premise" and decision implication canonical basis, and prove that decision implication canonical basis is complete and non-redundant, and moreover the basis contains the least number of decision implications among all complete sets of decision context.

Additionally, further research can be conducted from the following perspectives:

1. Just as mentioned, d-premise has recursive characteristic in nature, so how to compute d-premise efficiently is worth further studies. Here we note down two possible ways to improve Algorithm 1: (1) to characterize some properties of d-premise and merge lines 5–17 with line 4; and (2) to develop some efficient approach to generate minimal generators

(see [23,5,21]). On the other hand, we want to know whether or not the problem of checking whether a subset $P \subseteq C$ is a d-premise is NP-hard or, as in the case of pseudo-intent, coNP-complete;

2. By examining Example 6, we find that only one decision implication is minimal but not proper. It would be interesting to know how different decision implication canonical basis and the set of α-maximal decision implications are. Possibly, we conjecture, the difference might rely chiefly on number of decision attributes, i.e., the less the number of decision attributes, the more close the two sets, partially because less decision attributes will provide less chances to violate condition (2) of d-premise;

3. Interesting works also include how to extend decision implication canonical basis to the setting of information system [17] (or many-valued context in FCA [9]), to the decision case of association rules [1], and to the decision case of fuzzy concept lattice [3].

Acknowledgments

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