Fuzzy Granular Structure Distance

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***Abstract*—A fuzzy granular structure refers to a mathematical structure of the collection of fuzzy information granules granulated from a dataset, while a fuzzy information granularity is used to measure its uncertainty. However, the existing forms of fuzzy information granularity have two limitations. One is that when the fuzzy information granularity of one fuzzy granular structure equals that of the other, one can say that these two fuzzy granular structures possess the same uncertainty, but these two fuzzy granular structures may be not equivalent to each other. The other limitation is that existing axiomatic approaches to fuzzy information granularity are still not complete, under which when the partial order relation among fuzzy granular structures cannot be found, their coarseness/fineness relationships will not be revealed. To address these issues, a so-called fuzzy granular structure distance is proposed in this study, which can well discriminate the difference between any two fuzzy granular structures. Besides this advantage, the fuzzy granular structure distance has another important benefit: It can be used to establish a generalized axiomatic constraint for fuzzy information granularity. By using the axiomatic constraint, the coarseness/fineness of any two fuzzy granular structures can be distinguished. In addition, through taking the fuzzy granular structure distances of a fuzzy granular structure to the finest one and the coarsest one into account, we also can build a bridge between fuzzy information granularity and fuzzy information entropy. The applicable analysis on 12 real-world datasets shows that the fuzzy granular structure distance and the generalized fuzzy information granularity have much better performance than existing methods.**

***Index Terms*—Granular computing (GrC), fuzzy granular structure distance, fuzzy information entropy, fuzzy information granularity.**

I. Introduction

GRANULAR computing (GrC) was first proposed by Zadeh in 1996 [55] and is becoming an important issue in artificial intelligence and information processing [56]–[58]. As Zadeh pointed out, information granulation, organization, and causation are three key issues in GrC. It has been applied in various fields, which include data clustering, machine learning, approximate reasoning, data mining and knowledge discovery, and so on. To date, several methods have been employed for studying GrC, such as rough set theory [4], [10], [16], [39], [47], fuzzy set theory [25]–[27], [29], [30], [46], [50], [51], concept lattice theory [11], [28], [49], and quotient space theory [61].

Pawlak established the rough set theory in 1982 [31], [32], which can be seen as a new method for studying uncertainty [12]–[15], [40]–[43], [48], [52]. In the context of a rough set, a given equivalence relation divides a dataset into some classes or concepts, often called a granular structure in GrC, and an equivalence class is called an information granule [17], [18]. As a basic concept of rough set theory, a granular structure base means a family of granular structures, where each granular structure is induced by a crisp binary relation. The crisp binary relations include equivalence relation, tolerance relation, neighborhood relation, dominance relation, and so on. If we employ a fuzzy binary relation for granulating a dataset, objects will be granulated to a fuzzy granular structure, i.e., a collection of fuzzy information granules [3], [33]–[36], [41], which can be used to construct rough approximations of a fuzzy rough set [1], [2], [7]–[9], [54], [59], [60]. Similar to the concept of granular structure base, a fuzzy granular structure base correspondingly indicates a set of fuzzy granular structures induced by a family of fuzzy binary relations.

Information granularity is a measure to calculate the granulation degree of a universe in the GrC area. It has been an important problem of how to compute the information granularity of a granular structure in GrC. For fuzzy-set-based GrC, fuzzy information granularity is employed for measuring the granulation degree of a fuzzy granular structure induced by a given dataset. The smaller the fuzzy information granularity, the finer a fuzzy granular structure. Up to now, several definitions of (fuzzy) information granularity have been developed with various perspectives and viewpoints [12], [14], [15], [41], [48], [52]. Liang *et al.* [14], [15] contributed two forms of information granularity for measuring that of complete data and that of incomplete data, respectively. Wierman [48] gave a so-called granulation measure to evaluate the uncertainty of knowledge from a knowledge base, and its form is the same as Shannon entropy in some sense. Combination granulation proposed by Qian and Liang [12] can also be used to measure the granulation degree of knowledge from a knowledge base. Xu *et al.* [52] improved the roughness in rough set theory given by Pawlak [31], which also can be seen as an information granularity. Qian *et al.* [41] put forward two forms of fuzzy information granularity to measure the coarseness/fineness of a fuzzy knowledge structure. To obtain a constraint framework of fuzzy information granularity, a series of axiomatic approaches to fuzzy information

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granularity were developed in the literature [41]. For revealing the properties of information granularity, a partial order relation is often employed for depicting the monotonicity between granular structures. However, the fuzzy information granularity still has its shortages. In what follows, we analyze two limitations of the existing fuzzy information granularities, which become the main motivations of this study.

1) Usually, if the fuzzy information granularities of two fuzzy granular structures are equivalent, then one means that uncertainties of these two fuzzy granular structures are identical. However, we cannot judge that they are the same granular structure. That is to say, the fuzzy information granularity cannot well differentiate two fuzzy granular structures from the same fuzzy knowledge base.
2) An axiomatic constraint of fuzzy information granularity proposes constraints of how to define a reasonable measure for quantifying the information granularity of a fuzzy granular structure, in which a partial order relation plays a very important role. In recent years, several partial order relations have been developed on fuzzy granular structures, where the granulation partial order relation is the most successful for distinguishing the coarseness/fineness between two fuzzy granular structures. Despite its success, the partial order relation often cannot be found between many fuzzy granular structures. This shows that the existing axiomatic approaches still are incomplete for depicting axiomatic constraints of a fuzzy information granularity.

From the above these analyses, it can be seen that fuzzy information granularity still needs further study. To address these issues, in this paper, we first present a new concept, fuzzy granular structure distance, for differentiating two fuzzy granular structures from the same universe. Its some interesting properties are also analyzed, which are used to verify its correctness, validity, and rationality. Based on the fuzzy granular structure distance, one gives an axiomatic approach to fuzzy information granularity, called a generalized fuzzy information granularity (GFIG), which is established based on the fuzzy granular structure distance between a fuzzy granular structure and the finest one. This developed axiomatic approach can well overcome the limitation of existing versions. Finally, through using the fuzzy granular structure distance, we also build a bridge between fuzzy information granularity and fuzzy information entropy. This bridge shows that in some sense, there may be a complement relationship between the fuzzy information granularity and the fuzzy information entropy.

The organization of the rest of the paper is as follows. In Section II, several preliminary concepts in GrC are briefly recalled. In Section III, we discuss two limitations of existing forms of information granularity. To overcome these limitations, Section IV presents a so-called fuzzy granular structure distance to characterize the difference between any two fuzzy granular structures and gives its several interesting properties. In Section V, through analyzing existing axiomatic approaches to fuzzy information granularity, based on the proposed fuzzy granular structure distance, we develop a much more generalized axiomatic approach, called a GFIG, which solves the problem that each of existing partial order relations between fuzzy granular structures is often not found. In Section VI, we also built a bridge between fuzzy information granularity and fuzzy information entropy. Finally, Section VII gives a conclusion of this paper.

II. Preliminaries

In GrC, granular structure bases, fuzzy granular structure bases, fuzzy information granules, and fuzzy granular structures are several important concepts, which will be briefly reviewed in this section.

An approximation space $K = (U, R)$ in rough set theory is also called a granular structure in GrC, where U is a finite and nonempty set, called a universe, and $R \subseteq U \times U$ is an equivalence relation on U [31], [32]. The universe U can be partitioned into some disjoint classes by a given equivalence relation R, which is generally called a quotient set, just U/R. An equivalence relation is a special kind of similarities among objects from a dataset. When two objects are included in the same class $E_R(x)$, one can say that these two objects cannot be distinguished using the equivalence relation R. In general, a granular structure determined by R on U cam be formally represented as $F(R) = \{E_R(x) \mid x \in U\}$, in which each equivalence class $E_R(x),\ x \in U$, is viewed as an information granule consisting of indistinguishable objects [23], [24], [38]. A family of granular structures from the same universe is called a granular structure base, denoted by $F = (U, \mathbf{R})$, where U is a finite universe, and $\mathbf{R}$ is a set of equivalence relations.

Given a granular structure base $F = (U, \mathbf{R})$, one knows that every granular structure $F(R) = \{E_R(x) \mid x \in U\}$ is a cover of the universe U, where $\forall x \in U, E_R(x) \neq \emptyset$ and $\bigcup_{x\in U} E_R(x) = U$ hold. Given this representation, a partial order relation $\preceq$ has been introduced [12], [13], [37], [53], which is as follows:

$$P \preceq\ Q(P,\ Q \in \mathbf{R}) \Leftrightarrow E_P(x_i) \subseteq E_Q(x_i) \text{ for any } i \in\ \{1, 2, \ldots, |U|\}.$$

If $P \preceq Q$, one can say that P is much finer than Q. It has been proved $(\mathbf{R}, \preceq)$ is a poset [12], [53].

However, as Professor Zadeh pointed out, a crisp information granulation does not well characterize the fact that in much, perhaps most, the granules of human reasoning and information granulation are fuzzy rather than crisp [56]. It is necessary to generalize crisp information granulations to fuzzy cases. To address this issue, we review the following concepts in fuzzy cases.

In fuzzy information granulation, an equivalence relation in the crisp information granulation is replaced by a fuzzy binary relation $\widetilde{R}$ from a given universe U. We often represents a fuzzy binary relation by a relation matrix, which is formally as follows:

$$M(\widetilde{R}) = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix} \tag{1}$$

where $r_{ij} \in [0, 1]$ means the similarity between two objects x_i and x_j.

Given two fuzzy binary relations $\widetilde{R}_1, \widetilde{R}_2$, several operations between them have been often defined as

1) $\widetilde{R}_1 = \widetilde{R}_2 \Leftrightarrow \widetilde{R}_1(x,y) = \widetilde{R}_2(x,y)$, for all x, y;
2) $\widetilde{R} = \widetilde{R}_1 \cup \widetilde{R}_2 \Leftrightarrow \widetilde{R} = \max\{\widetilde{R}_1(x,y), \widetilde{R}_2(x,y)\}$;
3) $\widetilde{R} = \widetilde{R}_1 \cap \widetilde{R}_2 \Leftrightarrow \widetilde{R} = \min\{\widetilde{R}_1(x,y), \widetilde{R}_2(x,y)\}$;
4) $\widetilde{R}_1 \subseteq \widetilde{R}_2 \Leftrightarrow \widetilde{R}_1(x,y) \le \widetilde{R}_2(x,y)$, for all x, y.

Similar to an equivalence relation, given a universe, a fuzzy binary relation can correspondingly induce a set of fuzzy information granules, which is regarded as a fuzzy binary granular structure. In order to uniformly represent, the granulation result characterized by this family of fuzzy information granules is uniformly called a fuzzy granular structure in this paper. A fuzzy binary granular structure on U is formally written as

$$F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2), \ldots, G_{\widetilde{R}}(x_n)) \tag{2}$$

where $G_{\widetilde{R}}(x_i) = r_{i1}/x_1 + r_{i2}/x_2 + \cdots + r_{in}/x_n$. $G_{\widetilde{R}}(x_i)$ means the fuzzy information granule determined by x_i with respect to R, and r_{ij} is the similarity between objects x_i and x_j [3], [6]. Here, "+" indicates the union of objects. In fact, $G_{\widetilde{R}}(x_i)$ also can be understood as the fuzzy neighborhood of x_i in a sense. The cardinality of the fuzzy information granule $G_{\widetilde{R}}(x_i)$ can be calculated with

$$|G_{\widetilde{R}}(x_i)| = \sum_{j=1}^{n} r_{ij}. \tag{3}$$

A family of fuzzy binary granular structures is called a fuzzy granular structure base, denoted by $F = (U, \widetilde{\mathbf{R}})$. To uniformly represent granular structures, in this study, a fuzzy binary granular structure determined by $\widetilde{P} \in \widetilde{\mathbf{R}}$ is denoted as $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \cdots, G_{\widetilde{P}}(x_n))$, where $G_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i$. In this case, the granular structure is also a binary neighborhood system [17]–[24]. Furthermore, let $\mathbf{F}(U)$ denote the collection of all fuzzy binary granular structures from a given universe U.

Given a fuzzy binary granular structure $F = (S_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))$, in particular, if $p_{ij} = 0,\ i, j \le n$, then $|G_{\widetilde{P}}(x_i)| = 0, i \le n$, and the fuzzy granular structure is called the finest one, write as $\widetilde{P} = \widetilde{\omega}$, i.e., $F(\widetilde{\omega}) = (G_{\widetilde{\omega}}(x_1), G_{\widetilde{\omega}}(x_2), \ldots, G_{\widetilde{\omega}}(x_n))$, where $G_{\widetilde{\omega}}(x_i) = \sum_{j=1}^{n} \frac{\omega_{ij}}{x_j}, \forall i, j \le n, \omega_{ij} = 0$; if $p_{ij} = 1,\ i, j \le n$, then $|G_{\widetilde{P}}(x_i)| = |U|, i \le n$, and the fuzzy granular structure is called the coarsest one, write as $\widetilde{P} = \widetilde{\delta}$, i.e., $F(\widetilde{\delta})(G_{\widetilde{\delta}}(x_1), G_{\widetilde{\delta}}(x_2), \cdots, G_{\widetilde{\delta}}(x_n))$, where $G_{\widetilde{\delta}}(x_i) = \sum_{j=1}^{n} \frac{\delta_{ij}}{x_j}, \forall i, j \le n, \delta_{ij} = 1$.

These fuzzy granular structures found some base units in human fuzzy reasoning. The underlying algebra structure among $\mathbf{F}(U)$ has been discovered, which can used to reveal the hierarchical structure on fuzzy granular structures [41]. To investigate this issue, four operators among fuzzy granular structures have been proposed for revealing the algebra structure. These four operators in a family of fuzzy binary granular structures are defined by the following definition.

Definition 1: Let $\mathbf{F}(U)$ be the collection of all fuzzy binary granular structures on the universe U, $G(\widetilde{P}), G \in \mathbf{F}(U)$ two fuzzy granular structures. Four operators $\bigcap$, $\bigcup$, $-$, and $\wr$ on $\mathbf{F}(U)$ are defined as

$$\begin{aligned} F(\widetilde{P}) \bigcap F(\widetilde{Q}) &= \{G_{\widetilde{P}\cap\widetilde{Q}}(x_i) \mid G_{\widetilde{P}\cap\widetilde{Q}}(x_i) \\ &= G_{\widetilde{P}}(x_i) \cap G_{\widetilde{Q}}(x_i)\} \end{aligned} \tag{4}$$

$$\begin{aligned} F(\widetilde{P}) \bigcup F(\widetilde{Q}) &= \{G_{\widetilde{P}\cup\widetilde{Q}}(x_i) \mid G_{\widetilde{P}\cup\widetilde{Q}}(x_i) \\ &= G_{\widetilde{P}}(x_i) \cup G_{\widetilde{Q}}(x_i)\} \end{aligned} \tag{5}$$

$$\begin{aligned} F(\widetilde{P}) - F(\widetilde{Q}) &= \{G_{\widetilde{P}-\widetilde{Q}}(x_i) \mid G_{\widetilde{P}-\widetilde{Q}}(x_i) \\ &= G_{\widetilde{P}}(x_i) \cap \sim G_{\widetilde{Q}}(x_i)\} \end{aligned} \tag{6}$$

$$\wr F(\widetilde{P}) = \{\wr S_{\widetilde{P}}(x_i) \mid \wr G_{\widetilde{P}}(x_i) = \sim G_{\widetilde{P}}(x_i)\} \tag{7}$$

where $x_i \in U,\ i \le n$ and $\sim G_{\widetilde{P}}(x_i) = (1 - p_{i1})/x_i + (1 - p_{i2})/x_i + \cdots + (1 - p_{in})/x_i$.

These four operators are used to execute intersection operation, union operation, subtraction operation, and complement operation in-between fuzzy granular structures. Based on these four operators, we can fine, coarsen, decompose fuzzy granular structures and calculate complement of a fuzzy granular structure, respectively. It deserves to point out that $\bigcap$, $\bigcup$, $-$, and $\wr$ can be seen as four atomic formulas, and their finite connections are also formulas. In the context of these four operators, it has been proved that the algebra structure of these fuzzy granular structures is a lattice structure. In addition, those proposed four operators also can be employed for generating some new fuzzy granular structures on the same universe. That is to say, on the same universe, we can induce new fuzzy binary granular structures by some known fuzzy binary granular structures through combining these operators. Furthermore, these four operators have some nice properties, which have been discussed in [41].

III. Two Limitations of Fuzzy Information Granularity

Fuzzy information granularity and fuzzy information entropy are two main approaches to measuring the uncertainty of a fuzzy granular structure [3], [41]. A fuzzy information granularity is used to assess the coarseness of a fuzzy granular structure, while a fuzzy information entropy is adopted for measuring the uncertainty of the actual structure of a fuzzy granular structure. As Qian *et al.* pointed out [41], in a sense, the relationship between fuzzy information entropy and fuzzy information granularity may be a complement relationship, and they have the same capability for characterizing the uncertainty of a fuzzy binary granular structure. However, the existing definitions of fuzzy information granularity still have two shortages, which are revealed by the following two subsections, respectively.

A. First Limitation of Fuzzy Information Granularity

In GrC, the scale of each of information granules is often taken into account for designing measures of information granularity [14], [33]–[36], [38], which are used to compute the degree of granulation of a crisp granular structure. Some of fuzzy information entropies are also defined based on sizes of fuzzy information granules in a fuzzy granular structure. To measure the information granularity of a fuzzy granular structure, the literature [3] and the literature [41] developed two forms of fuzzy information granularities, respectively. In the following, we only review these two definitions of fuzzy information granularity.

Definition 2: Let $F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2), \ldots, G_{\widetilde{R}}(x_n))$. Then, fuzzy information granularity of $\widetilde{R}$ is defined as

$$GK(\widetilde{R}) = \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{R}}(x_i)|}{n} \tag{8}$$

where $|G_{\widetilde{R}}(x_i)|$ is the cardinality of the fuzzy information granule $G_{\widetilde{R}}(x_i)$.

Definition 3: Let $F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2), \ldots, G_{\widetilde{R}}(x_n))$. Then, fuzzy information granularity of $\widetilde{R}$ is defined as

$$E_r(\widetilde{R}) = -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|G_{\widetilde{R}}(x_i)|} \tag{9}$$

where $|G_{\widetilde{R}}(x_i)|$ is the cardinality of the fuzzy information granule $G_{\widetilde{R}}(x_i)$.

Usually, if the fuzzy information granularity (or fuzzy information entropy) of one fuzzy granular structure is equal to that of the other fuzzy granular structure, one can say that these two fuzzy granular structures possess the same uncertainty. However, this does not mean that these two fuzzy granular structures are equivalent each other. That is to say, the fuzzy information entropy and the fuzzy information granularity cannot well reveal the difference between two fuzzy granular structures in a fuzzy granular structure base. It can be seen from the following examples.

Example 1: Let $U = \{x_1, x_2, x_3\}$, $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1)G_{\widetilde{P}}, (x_2), G_{\widetilde{P}}(x_3)) \in \mathbf{F}(U)$ and $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1)G_{\widetilde{Q}}(x_2), G_{\widetilde{Q}}(x_3)) \in \mathbf{F}(U)$ be two fuzzy granular structures, where $G_{\widetilde{P}}(x_1) = 0.2/x_1 + 0.3/x_2 + 0.6/x_3$, $G_{\widetilde{P}}(x_2) = 0.4/x_2 + 0.7/x_2 + 0.8/x_2$, $G_{\widetilde{P}}(x_3) = 0.2/x_1 + 0.2/x_2 + 0.6/x_3$, and $G_{\widetilde{Q}}(x_1) = 0.4/x_1 + 0.4/x_2 + 0.3/x_3$, $G_{\widetilde{Q}}(x_2) = 0.3/x_1 + 0.4/x_2 + 0.3/x_3$, $G_{\widetilde{Q}}(x_3) = 0.5/x_1 + 0.6/x_2 + 0.8/x_3$.

From Definition 2, we calculate their fuzzy information granularities as follows:

$$GK(\widetilde{P}) = \frac{1}{3}\sum_{i=1}^{3}\frac{|G_{\widetilde{P}}(x_i)|}{3} = \frac{4}{9}$$

$$GK(\widetilde{Q}) = \frac{1}{3}\sum_{i=1}^{3}\frac{|G_{\widetilde{P}}(x_i)|}{3} = GK(\widetilde{P}) = \frac{4}{9}.$$

That is $GK(\widetilde{P}) = GK(\widetilde{Q})$.

From Definition 3, we compute the information granularities of these two fuzzy granular structures as follows:

$$E_r(\widetilde{P}) = -\sum_{i=1}^{3}\frac{1}{3}\log_2\frac{1}{|G_{\widetilde{R}}(x_i)|} = \frac{1}{3}\log_2 2.09$$

$$E_r(\widetilde{Q}) = -\sum_{i=1}^{3}\frac{1}{3}\log_2\frac{1}{|G_{\widetilde{R}}(x_i)|} = \frac{1}{3}\log_2 2.09.$$

That is, $E_r(\widetilde{P}) = E_r(\widetilde{Q})$.

However, the fuzzy granular structure $F(\widetilde{P})$ is clearly not equal to $F(\widetilde{Q})$. It shows that the fuzzy information granularity cannot effectively differentiate any two fuzzy granular structures. Fuzzy information entropy also has the same shortage, and hence, we omit its discussion here.

B. Second Limitation of Fuzzy Information Granularity

For characterizing the uncertainty of a granular structure, a partial order relation plays a very important role. In recent years, several partial order relations on fuzzy granular structures have been developed. In what follows, we review the existing partial order relations and their properties.

In what follows, we suppose $F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, where $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))G_{\widetilde{P}}(x_i) = p_{i1}/x_1, + \cdots + p_{ii}/x_i + \cdots + p_{in}/x_n$, $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1), G_{\widetilde{Q}}(x_2), \ldots, G_{\widetilde{Q}}(x_n))$, and $G_{\widetilde{Q}}(x_i) = q_{i1}/x_1 + \cdots + q_{ii}/x_i + \cdots + q_{in}/x_n$; then, the existing partial order relations and their properties are as follows.

The partial order relation $\widetilde{\preceq}_1$ is defined as [3], [45]:

$F(\widetilde{P})\widetilde{\preceq}_1 F(\widetilde{Q}) \Leftrightarrow G_{\widetilde{P}}(x_i) \subseteq G_{\widetilde{Q}}(x_i)$, for all $i \le n \Leftrightarrow p_{ij} \le q_{ij}$, for all $i, j \le n$, just $\widetilde{P}\widetilde{\preceq}_1\widetilde{Q}$. It is called a rough partial order relation.

Furthermore, $F(\widetilde{P}) = F(\widetilde{Q}) \Leftrightarrow G_{\widetilde{P}}(x_i) = G_{\widetilde{Q}}(x_i)$, for all $i \le n \Leftrightarrow p_{ij} = q_{ij}$, for all $i, j \le n$, write as $\widetilde{P} = \widetilde{Q}$. $F(\widetilde{P})\widetilde{\prec}_1 F(\widetilde{Q}) \Leftrightarrow F(\widetilde{P})\widetilde{\preceq}_1 F(\widetilde{Q})$ and $F(\widetilde{P}) \ne F(\widetilde{Q})$, denoted by $\widetilde{P}\widetilde{\prec}_1\widetilde{Q}$.

The partial order relation $\widetilde{\preceq}_2$ is defined as [41]: $F(\widetilde{P})\widetilde{\preceq}_2 F(\widetilde{Q}) \Leftrightarrow |G_{\widetilde{P}}(x_i)| \le |G_{\widetilde{Q}}(x_i)|$, for all $i \le n$, where $|G_{\widetilde{P}}(x_i)| = \sum_{j=1}^{n} p_{ij}$, $|G_{\widetilde{Q}}(x_i)| = \sum_{j=1}^{n} q_{ij}$, just $\widetilde{P}\widetilde{\preceq}_2\widetilde{Q}$. The partial order relation is called a generalized rough partial order relation.

Moreover, $F(\widetilde{P}) \simeq F(\widetilde{Q}) \Leftrightarrow |G_{\widetilde{P}}(x_i)| = |G_{\widetilde{Q}}(x_i)|$, for all $i \le n$, just $\widetilde{P} \simeq \widetilde{Q}$. $F(\widetilde{P})\widetilde{\prec}_2 F(\widetilde{Q}) \Leftrightarrow F(\widetilde{P})\widetilde{\preceq}_2 F(\widetilde{Q})$ and $F(\widetilde{P}) \not\simeq F(\widetilde{Q})$, write as $\widetilde{P}\widetilde{\prec}_2\widetilde{Q}$.

The partial order relation $\preceq_3$ is defined as [41]:

$F(\widetilde{P})\widetilde{\preceq}_3 F(\widetilde{Q}) \Leftrightarrow$ for $F(\widetilde{P})$, there exists a sequence $F'(\widetilde{Q})$ of $F(\widetilde{Q})$ such that $|G_{\widetilde{P}}(x_i)| \le |G_{\widetilde{Q}}(x_i^{'})|$, for all $i \le n$, just $\widetilde{P}\widetilde{\preceq}_3\widetilde{Q}$, where $F'(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1^{'}), G_{\widetilde{Q}}(x_2^{'}), \ldots, G_{\widetilde{Q}}(x_n^{'}))$. It is called a granulation partial order relation.

In addition, $F(\widetilde{P}) \approx F(\widetilde{Q}) \Leftrightarrow |G_{\widetilde{P}}(x_i)| = |G_{\widetilde{Q}}(x_i^{'})|$, for all $i \le n$, denoted by $\widetilde{P} \approx \widetilde{Q}$. $F(\widetilde{P})\widetilde{\prec}_3 F(\widetilde{Q}) \Leftrightarrow F(\widetilde{P})\widetilde{\preceq}_3 F(\widetilde{Q})$ and $F(\widetilde{P}) \not\approx F(\widetilde{Q})$, write as $\widetilde{P}\widetilde{\prec}_3\widetilde{Q}$.

To date, these three partial order relations have been well used to compare the coarseness/fineness between two given fuzzy granular structures from the same universe. The relationships among these three partial order relations had been established with the following three theorems.

Theorem 1 (see[41]): Partial order relation $\widetilde{\preceq}_1$ is a special instance of partial relation $\widetilde{\preceq}_2$.

Theorem 2 (see[41]): Partial order relation $\widetilde{\preceq}_2$ is a special instance of partial relation $\widetilde{\preceq}_3$.

Theorem 3 (see[41]): Partial order relation $\preceq_1$ is a special instance of partial relation $\widetilde{\preceq}_3$.

From the above theorems, one can draw such a conclusion that the partial order relation $\widetilde{\preceq}_3$ is the best one for distinguishing the coarseness/fineness between two fuzzy granular structures. However, the partial order relation $\widetilde{\preceq}_3$ still has its shortages for distinguishing fuzzy granular structures. This is because one cannot find these partial order relations among some fuzzy granular structures, which is illustrated with Example 2.

Example 2: Let $U = \{x_1, x_2, x_3, x_4\}$, $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), G_{\widetilde{P}}(x_3), G_{\widetilde{P}}(x_4)) \in \mathbf{F}(U)$ and $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1), G_{\widetilde{Q}}(x_2), G_{\widetilde{Q}}(x_3), G_{\widetilde{Q}}(x_4)) \in \mathbf{F}(U)$ be two fuzzy granular structures, where $G_{\widetilde{P}}(x_1) = 1/x_1 + 0/x_2 + 0/x_3 + 0/x_4$, $G_{\widetilde{P}}(x_2) = 0.3/x_2 + 0.6/x_2 + 0/x_2 + 0/x_4$, $G_{\widetilde{P}}(x_3) = 0/x_1 +0/x_2 + 0.4/x_3 + 0/x_4$, $G_{\widetilde{P}}(x_4) = 0/x_1 + 0/x_2 + 0/x_3 + 0.1/x_4$, and $G_{\widetilde{Q}}(x_1) = 1/x_1 + 0.6/x_2 + 0/x_3 + 0.7/x_4$, $G_{\widetilde{Q}}(x_2) = 0.3/x_1 + 0.7/x_2 + 0.8/x_3 + 0/x_4$, $G_{\widetilde{Q}}(x_3) = 0/x_1 + 0/x_2 + 0/x_3 + 0/x_4$, $G_{\widetilde{Q}}(x_4) = 0/x_1 + 0/x_2 + 0.7/x_3 + 0.4/x_4$.

For this example, there does not exist any array of members in $F(\widetilde{Q})$ such that $F(\widetilde{P})\widetilde{\preceq}_3 F(\widetilde{Q})$ or $F(\widetilde{Q})\widetilde{\preceq}_3 F(\widetilde{P})$. Nevertheless, the fuzzy granular structure $F(\widetilde{Q})$ should be much coarser than the fuzzy granular structure $F(\widetilde{P})$, intuitively. Unfortunately, one cannot differentiate the coarseness/fineness between these two fuzzy granular structures through using the granulation partial order relation $\widetilde{\preceq}_3$ in this case. That is to say, when there does not exist one of these three partial order relations between $F(P)$ and $F(Q)$, their information granularities cannot be compared. Hence, the axiomatic definitions of information granularity based on these partial order relations still have such a limitation for characterizing coarseness/fineness degrees among fuzzy granular structures.

From Section III-A and III-B, the existing forms of fuzzy information granularity have two obvious limitations, which brings a challenge for studying uncertainty in GrC. To overcome these limitations, it is very desirable to develop a measure for differentiating two fuzzy granular structures, which is an important problem in GrC.

IV. Fuzzy Granular Structure Distance and Its Properties

In this section, we will introduce a concept of fuzzy granular structure distance to distinguish two given fuzzy knowledge structures.

From the composition of a fuzzy granular structure, fuzzy information granules are basic units. To give an effective distance between two fuzzy granular structures, the fuzzy information granules determined by an object with two fuzzy binary relations should be well differentiated. The accumulation of differences on fuzzy information granules determined by all objects can characterize the entire difference between two fuzzy granular structures from the same universe.

Based on the above idea, given a universe U, we introduce a new concept of fuzzy granular structure distance with the following definition.

Definition 4: Let $F = (U, \widetilde{\mathbf{R}})$ be a fuzzy granular structure base, $\widetilde{P}, \widetilde{Q} \in \widetilde{\mathbf{R}}$, $F(\widetilde{P}) = \{G_{\widetilde{P}}(x),\ x \in U\}$ and $F(\widetilde{Q}) = \{G_{\widetilde{Q}}(x),\ x \in U\}$ two fuzzy granular structures. The fuzzy granular structure distance between $F(\widetilde{P})$ and $F(\widetilde{Q})$ is formally defined as

$$D(F(\widetilde{P}), F(\widetilde{Q})) = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{Q}}(x_i)|}{|U|} \tag{10}$$

where $|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{Q}}(x_i)| = |G_{\widetilde{P}}(x_i) \cup G_{\widetilde{Q}}(x_i)| - |G_{\widetilde{P}}(x_i) \cap G_{\widetilde{Q}}(x_i)|$, $x_i \in U$.

The fuzzy granular distance can well describe the difference between two fuzzy granular structures coming from the same universe.

Theorem 4 (Extremum): Let $\mathbf{F}(U)$ be the collection of all fuzzy granular structures induced by the universe U, $F(\widetilde{P})$, $F(\widetilde{Q})$ two granular structures in $\mathbf{F}(U)$. Then, $D(F(\widetilde{P}), F(\widetilde{Q}))$ achieves its minimum value $D(F(\widetilde{P}), F(\widetilde{Q})) = 0$ if and only if $F(\widetilde{P}) = F(\widetilde{Q})$; and $D(F(\widetilde{P}), F(\widetilde{Q}))$ achieves its maximum value 1 if $\widetilde{P} = \omega$ and $\widetilde{Q} = \delta$ (or $\widetilde{P} = \delta$ and $\widetilde{Q} = \omega$).

Obviously, $0 \leq D(F(\widetilde{P}), F(\widetilde{Q})) \leq 1$ holds.

In what follows, we continue to employ Example 1 for verifying the validity of the fuzzy granular structure distance.

Example 3 (Continued from Example 1): By Definition 4, it follows that

$$\begin{aligned} D(F(\widetilde{P}), F(\widetilde{Q})) &= \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{Q}}(x_i)|}{|U|} \\ &= \frac{1}{3}\left(\frac{0.6 + 0.9 + 0.9}{3}\right) = \frac{2.4}{9}. \end{aligned}$$

It can be seen that the fuzzy granular structure distance can effectively measure the difference of those two fuzzy granular structures in Example 1.

In what follows, we investigate some of important properties of the fuzzy granular structure distance proposed above.

Based on the definition of the fuzzy rough partial relation $\widetilde{\preceq}_1$ among fuzzy granular structures, we can find that the relation among fuzzy granular structures is based on the inclusion relations between two fuzzy information granules of every object with two fuzzy binary relations. Therefore, we can employ the rough partial relation $\widetilde{\preceq}_1$ for investigating the properties of the fuzzy granular structure distance.

For further investigation, we first give a distance between two fuzzy sets with the same number of objects.

Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy sets; then, the difference between them can be described by the equation as follows:

$$d(\widetilde{A}, \widetilde{B}) = |\widetilde{A} \cup \widetilde{B}| - |\widetilde{A} \cap \widetilde{B}|. \tag{11}$$

For the distance between fuzzy sets, one can obtain the following lemma.

Lemma 1: Let $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ be three fuzzy sets on the same universe, $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$ or $\widetilde{A} \supseteq \widetilde{B} \supseteq \widetilde{C}$; then, $d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C}) = d(\widetilde{A}, \widetilde{C})$.

Proof: Supposing $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, then for any $x_i \in U$, we have $\mu_{\widetilde{A}}(x_i) \leq \mu_{\widetilde{B}}(x_i) \leq \mu_{\widetilde{C}}(x_i)$. Hence

$$\begin{aligned} &d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C}) \\ &= |\widetilde{A} \cup \widetilde{B}| - |\widetilde{A} \cap \widetilde{B}| + |\widetilde{B} \cup \widetilde{C}| - |\widetilde{B} \cap \widetilde{C}| \end{aligned}$$

$$= \sum_{i=1}^{n} \mu_{\widetilde{B}}(x_i) - \sum_{i=1}^{n} \mu_{\widetilde{A}}(x_i) + \sum_{i=1}^{n} \mu_{\widetilde{C}}(x_i) - \sum_{i=1}^{n} \mu_{\widetilde{B}}(x_i)$$

$$= \sum_{i=1}^{n} \mu_{\widetilde{C}}(x_i) - \sum_{i=1}^{n} \mu_{\widetilde{A}}(x_i)$$

$$= |\widetilde{A} \cup \widetilde{C}| - |\widetilde{A} \cap \widetilde{C}|$$

$$= d(\widetilde{A}, \widetilde{C}).$$

If $\widetilde{A} \supseteq \widetilde{B} \supseteq \widetilde{C}$, similarly, we have $d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C}) = d(\widetilde{A}, \widetilde{C})$. This completes the proof. ■

Let $F(\widetilde{P}) = \{G_{\widetilde{P}}(x),\ x \in U\}$, $F(\widetilde{Q}) = \{G_{\widetilde{Q}}(x),\ x \in U\}$, and $F(\widetilde{R}) = \{G_{\widetilde{R}}(x),\ x \in U\}$ be three fuzzy granular structures on the universe U. By Definition 4 and Lemma 1, we can get some theorems as follows.

Theorem 5: Let $F = (U, \widetilde{\mathbf{R}})$ be a fuzzy granular structure base, $\widetilde{P}$, $\widetilde{Q}$, $\widetilde{R} \in \widetilde{\mathbf{R}}$. If $F(\widetilde{P}) \widetilde{\preceq}_1 F(\widetilde{Q}) \widetilde{\preceq}_1 F(\widetilde{R})$ or $F(\widetilde{R}) \widetilde{\preceq}_1 F(\widetilde{Q}) \widetilde{\preceq}_1 F(\widetilde{P})$, then $D(F(\widetilde{P}), F(\widetilde{R})) = D(F(\widetilde{P}), F(\widetilde{Q})) + D(F(\widetilde{Q}), F(\widetilde{R}))$.

Proof: Suppose that $F(\widetilde{P}) \widetilde{\preceq}_1 F(\widetilde{Q}) \widetilde{\preceq}_1 F(\widetilde{R})$, for any $x_i \in U$, we have $G_{\widetilde{P}}(x_i) \subseteq G_{\widetilde{Q}}(x_i) \subseteq G_{\widetilde{R}}(x_i)$. By Lemma 1, one has that

$$D(F(\widetilde{P}), F(\widetilde{Q})) + D(F(\widetilde{Q}), F(\widetilde{R}))$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|G_{\widetilde{P}}(x_i) \Delta G_{\widetilde{Q}}(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|G_{\widetilde{Q}}(x_i) \Delta G_{\widetilde{R}}(x_i)|}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{P}}(x_i), G_{\widetilde{Q}}(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{Q}}(x_i), G_{\widetilde{R}}(x_i))}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{P}}(x_i), G_{\widetilde{Q}}(x_i)) + d(G_{\widetilde{Q}}(x_i), G_{\widetilde{R}}(x_i))}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{P}}(x_i), G_{\widetilde{R}}(x_i))}{|U|}$$

$$= D(F(\widetilde{P}), F(\widetilde{R})).$$

Similarly, when $F(\widetilde{R}) \widetilde{\preceq}_1 F(\widetilde{Q}) \widetilde{\preceq}_1 F(\widetilde{P})$, one also has that $D(F(\widetilde{P}), F(\widetilde{R})) = D(F(\widetilde{P}), F(\widetilde{Q})) + D(F(\widetilde{Q}), F(\widetilde{R}))$. This completes the proof. ■

This theorem is clearly illustrated by the following Example 4.

Example 4: Let $U = \{x_1, x_2\}$, $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1) G_{\widetilde{P}}(x_2))$, $\in \mathbf{F}(U)$, $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1),\ \ G_{\widetilde{Q}}(x_2)) \in \mathbf{F}(U)$, $F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2)) \in \mathbf{F}(U)$, and $F(\widetilde{P}) \widetilde{\preceq}_1 F(\widetilde{Q}) \widetilde{\preceq}_1 F(\widetilde{R})$, where $G_{\widetilde{P}}(x_1) = 0.1/x_1 + 0.2/x_2$, $G_{\widetilde{P}}(x_2) = 0.2/x_2 + 0.3/x_2$, $G_{\widetilde{Q}}(x_1) = 0.2/x_1 + 0.3/x_2$, $G_{\widetilde{Q}}(x_2) = 0.3/x_1 + 0.4/x_2$, $G_{\widetilde{R}}(x_1) = 0.3/x_1 + 0.4/x_2$, $G_{\widetilde{R}}(x_2) = 0.4/x_1 + 0.6/x_2$. By Definition 4, one can get $D(F(\widetilde{P}), F(\widetilde{Q})) = \frac{0.4}{4}$, $D(F(\widetilde{Q}).F(\widetilde{R})) = \frac{0.5}{4}$, and $D(F(\widetilde{Q}), F(\widetilde{R})) = \frac{0.9}{4}$; hence, $D(F(\widetilde{P}), F(\widetilde{R})) = D(F(\widetilde{P}), F(\widetilde{Q})) + D(F(\widetilde{Q}), F(\widetilde{R}))$.

From the above discussions and analysis, we can get three corollaries as follows.

Corollary 1: Let $\mathbf{F}(U)$ be a family of all fuzzy granular structures induced by a given universe U, $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$ two fuzzy granular structures. If $F(\widetilde{P}) \widetilde{\preceq}_1 F(\widetilde{Q})$, then one has that $D(F(\widetilde{P}), F(\widetilde{\omega})) \leq D(F(\widetilde{Q}), F(\widetilde{\omega}))$.

Corollary 2: Let $\mathbf{F}(U)$ be a family of all fuzzy granular structures induced by a given universe U, $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$ two fuzzy granular structures. If $F(\widetilde{P}) \widetilde{\preceq}_1 F(\widetilde{Q})$, then one has that $D(F(\widetilde{P}), F(\widetilde{\delta})) \geq D(F(\widetilde{Q}), F(\widetilde{\delta}))$.

In what follows, we discuss the triangle inequality of the fuzzy granular structure distance on $\widetilde{\mathbf{F}}(U)$.

Due to the maximum and minimum operators of the fuzzy set and (11), we can easily obtain another lemma as follows.

Lemma 2: Given three fuzzy sets $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, $d(\widetilde{A}, \widetilde{B}) + d(\widetilde{B}, \widetilde{C}) \geq d(\widetilde{A}, \widetilde{C})$, $d(\widetilde{A}, \widetilde{B}) + d(\widetilde{A}, \widetilde{C}) \geq d(\widetilde{B}, \widetilde{C})$, and $d(\widetilde{A}, \widetilde{C}) + d(\widetilde{B}, \widetilde{C}) \geq d(\widetilde{A}, \widetilde{B})$.

Based on the lemma above, one can draw a conclusion that $(\mathbf{F}(U), D)$ is a distance metric on $\mathbf{F}(U)$.

Theorem 6: Let $\mathbf{F}(U)$ be a family of all fuzzy granular structures induced by a given universe U; then, $(\mathbf{F}(U), D)$ is a distance space. ■

Proof:

1) By Definition 4, it is clear that $D(F(\widetilde{P}), K(\widetilde{Q})) \geq 0$.
2) From the symmetry of the operator Δ, one has that $D(F(\widetilde{P}), F(\widetilde{Q})) = D(F(\widetilde{Q}), K(\widetilde{P}))$.
3) In order to prove the triangle inequality, given three fuzzy granular structures $F(P)$, $F(Q)$ and $F(\widetilde{R}) \in \mathbf{F}(U)$, without loss of generality, one needs to prove $D(F(\widetilde{P}), F(\widetilde{Q})) + D(F(\widetilde{P}), F(\widetilde{R})) \geq D(F(\widetilde{Q}), F(\widetilde{R}))$.

By Lemma 2, for $x_i \in U$, $D(G_{\widetilde{P}}(x_i), G_{\widetilde{Q}}(x_i)) + D(G_{\widetilde{P}}(x_i), G_{\widetilde{R}}(x_i)) \geq D(G_{\widetilde{Q}}(x_i), G_{\widetilde{R}}(x_i))$; hence,

$$D(F(\widetilde{P}), K(\widetilde{Q})) + D(F(\widetilde{P}), K(\widetilde{R}))$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|G_{\widetilde{P}}(x_i) \Delta G_{\widetilde{Q}}(x_i)|}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{|G_{\widetilde{P}}(x_i) \Delta G_{\widetilde{R}}(x_i)|}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{P}}(x_i), G_{\widetilde{Q}}(x_i))}{|U|} + \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{P}}(x_i), G_{\widetilde{R}}(x_i))}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{1}{|U|} (d(G_{\widetilde{P}}(x_i), G_{\widetilde{Q}}(x_i)) + d(G_{\widetilde{P}}(x_i), G_{\widetilde{R}}(x_i)))$$

$$\geq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{d(G_{\widetilde{Q}}(x_i), G_{\widetilde{R}}(x_i))}{|U|}$$

$$= \frac{1}{|U|} \sum_{i=1}^{|U|} D(F(\widetilde{Q}), F(\widetilde{R})).$$

Analogously, one has that $D(F(\widetilde{R}), F(\widetilde{Q})) + D(F(\widetilde{P}), F(\widetilde{R})) \geq D(F(\widetilde{Q}), F(\widetilde{P}))$ and $D(F(\widetilde{R}), F(\widetilde{Q})) + D(F(\widetilde{P}), F(\widetilde{Q})) \geq D(F(\widetilde{R}), F(\widetilde{P}))$.

Therefore, $(\widetilde{\mathbf{F}}(U), D)$ is a distance space. ■

Example 5 (Continued from Example 2): By Definition 4, we can obtain that $D(K(\widetilde{P}), F(\widetilde{Q})) = \frac{2.6}{9}$, $D(F(\widetilde{Q}), F(\widetilde{R})) = \frac{2.6}{9}$, and $D(F(\widetilde{P}), F(\widetilde{R})) = \frac{1.4}{9}$. Thus, one has that $D(F(\widetilde{R}), F(\widetilde{Q})) + D(F(\widetilde{P}), \quad F(\widetilde{R})) \geq D(F(\widetilde{Q}), F(\widetilde{P}))$, $D(F(\widetilde{R}), F(\widetilde{Q})) + D(F(\widetilde{P}), F(\widetilde{Q})) \geq D(F(\widetilde{R}), F(\widetilde{P}))$.

From the above discussions, we conclude that the fuzzy granular structure distance is an effective metric for calculating the difference between two fuzzy granular structures from the same universe, which also can describe the geometric structure of all fuzzy granular structures from the same universe from the idea of geometry.

V. Generalized Fuzzy Information Granularity

In recent years, several researchers have already paid attention to the problem of what is the essence of fuzzy information granularity for fuzzy granular structures. Qian *et al.* [41] attempted to unify the definitions by using some existing axiomatic approaches to fuzzy information granularity. In this section, based on the proposed fuzzy granular structure distance, we aim to propose a generalized axiomatic definition to fuzzy information granularity.

Through employing the partial order relation $\preceq_i$, $i \in \{1, 2, 3\}$, Qian *et al.* [41] had given three axiomatic definitions of a fuzzy information granularity in the context of fuzzy binary granular structures.

Definition 5 (see[41]): Let $\mathbf{F}(U)$ be the set constructed by all fuzzy binary granular structures on the universe U. $\forall F(\widetilde{P}) \in \mathbf{F}(U)$, there exists a real number $g(\widetilde{P})$ satisfying the following properties:

1) $g(\widetilde{P}) \geq 0$ (Nonnegativity);
2) if $F(\widetilde{P}) = F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) = g(\widetilde{Q})$ (Invariability);
3) if $F(\widetilde{P}) \precsim_1 F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) < g(\widetilde{Q})$ (Monotonicity);

then g is called a fuzzy rough granularity (just FRG).

Definition 6 (see[41]): Let $\mathbf{F}(U)$ be the set constructed by all fuzzy binary granular structures on the universe U. $\forall F(\widetilde{P}) \in \mathbf{F}(U)$, there exists a real number $g(\widetilde{P})$ satisfying the following properties:

1) $g(\widetilde{P}) \geq 0$ (Nonnegativity);
2) if $F(\widetilde{P}) \simeq F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) = g(\widetilde{Q})$ (Invariability);
3) if $F(\widetilde{P}) \precsim_2 F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) < g(\widetilde{Q})$ (Monotonicity);

then g is called a generalized fuzzy rough granularity (just GFRG).

Definition 7 (see[41]): Let $\mathbf{F}(U)$ be the set constructed by all fuzzy binary granular structures on the universe U. $\forall F(\widetilde{P}) \in \mathbf{F}(U)$, there exists a real number $g(\widetilde{P})$ satisfying the following properties:

1) $g(\widetilde{P}) \geq 0$ (Nonnegativity);
2) if $F(\widetilde{P}) \approx F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) = g(\widetilde{Q})$ (Invariability);
3) if $F(\widetilde{P}) \precsim_3 F(\widetilde{Q})$, $\forall F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$, then $g(\widetilde{P}) < g(\widetilde{Q})$ (Monotonicity);

then g is called a fuzzy information granularity (just FIG).

For the above three axiomatic definitions of fuzzy information granularity, to date, the fuzzy information granularity has the strongest ability for differentiating the coarseness/fineness degrees of fuzzy granular structures. It is very interesting that the fuzzy granular structure distance can be used to construct a fuzzy information granularity. This mechanism is shown in the following theorem.

Theorem 7: Let $\mathbf{F}(U)$ be the set constructed by all fuzzy binary granular structures on the universe U. $\forall F(\widetilde{P}), F(\widetilde{\omega}) \in \mathbf{F}(U)$. Then, $D(F(\widetilde{P}), F(\widetilde{\omega}))$ is a fuzzy information granularity.

Proof: Assume U be a finite universe, let $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))$ and $F(\widetilde{\omega}) = (G_{\widetilde{\omega}}(x_1), G_{\widetilde{\omega}}(x_2), \ldots, G_{\widetilde{\omega}}(x_n))$, where $G_{\widetilde{\omega}}(x_i) = \sum_{j=1}^{n} \frac{\omega_{ij}}{x_j}, \forall i, j \leq n$, $\omega_{ij} = 0$.

1) Clearly, the distance D is nonnegative.
2) If $F(\widetilde{P}) \approx F(\widetilde{Q})$, then there must exist a bijective mapping function $f : F(\widetilde{P}) \to F(\widetilde{Q})$ such that $|G_{\widetilde{P}}(x_i)| = |f(G_{\widetilde{P}}(x_i))|$, $x_i \in U$, and $f(G_{\widetilde{P}}(x_i)) = G_{\widetilde{Q}}(x_{j_i})$. One has that

$$D(F(\widetilde{P}), F(\widetilde{\omega})) = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$$
$$= \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)| - 0}{|U|} = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|f(G_{\widetilde{P}}(x_i))| - 0}{|U|}$$
$$= \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_{j_i})| - 0}{|U|} = \frac{1}{|U|}\sum_{j=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_j)| - 0}{|U|}$$
$$= D(F(\widetilde{Q}), F(\widetilde{\omega})).$$

3) Now one proves that if $F(\widetilde{P}) \precsim_3 F(\widetilde{Q})$, then $D(F(\widetilde{P}), F(\widetilde{\omega})) < D(F(\widetilde{Q}), F(\widetilde{\omega}))$. Let $\widetilde{P}, \widetilde{Q} \in \widetilde{\mathbf{R}}$ with $F(\widetilde{P}) \precsim_3 F(Q)$, $F(\widetilde{P}) = \{G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_{|U|})\}$ and $F(\widetilde{Q}) = \{G_{\widetilde{Q}}(x_1), G_{\widetilde{Q}}(x_2), \ldots, G_{\widetilde{Q}}(x_{|U|})\}$, then there exists a sequence $F'(\widetilde{Q})$ of $F(\widetilde{Q})$, where $F'(\widetilde{Q}) = \{G_{\widetilde{Q}}(x_1'), G_{\widetilde{Q}}(x_2'), \ldots, G_{\widetilde{Q}}(x_{|U|}')\}$, such that $|G_{\widetilde{P}}(x_i)| \leq |G_{\widetilde{Q}}(x_i')|$, and there at least exists $x_s \in U$ such that $|G_{\widetilde{P}}(x_s)| < |f(G_{\widetilde{P}}(x_s))| = |G_{\widetilde{Q}}(x_s')|$. Thus

$$D(F(\widetilde{P}), F(\widetilde{\omega})) = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$$
$$= \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_P(x_i)| - 0}{|U|}$$

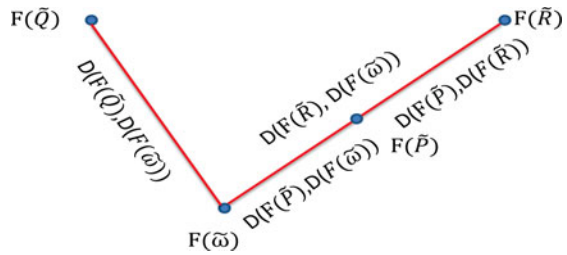

Fig. 1. Fuzzy granular structure distance with $F(\widetilde{\omega})$.

$$= \frac{1}{|U|}\left(\sum_{i=1,i\neq s}^{|U|}\frac{|G_{\widetilde{P}}(x_i)|-0}{|U|}+\frac{|G_{\widetilde{P}}(x_s)|-0}{|U|}\right)$$

$$< \frac{1}{|U|}\left(\sum_{i=1,i\neq s}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)|-0}{|U|}+\frac{|G_{\widetilde{Q}}(x_s')|-0}{|U|}\right)$$

$$= \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$$

$$= D(F(\widetilde{Q}),F(\widetilde{\omega})),$$

i.e., $D(F(\widetilde{P}),F(\widetilde{\omega})) < D(F(\widetilde{Q}),F(\widetilde{\omega}))$.

Summarizing above, $D(F(\widetilde{P}),F(\widetilde{\omega}))$ is a fuzzy information granularity. ■

From the theorem above, we can see that the fuzzy granular structure distance between the fuzzy granular structure $F(\widetilde{P})$ and the finest one $F(\widetilde{\omega})$ can be regarded as a fuzzy information granularity. In fact, the distance $D(F(\widetilde{P}),F(\widetilde{\omega}))$ has some better properties for depicting the information granularity of any fuzzy granular structure. Its advantages can be further explained in the following paragraph.

Through analyzing the sematic of the fuzzy granular structure distance $D(F(\widetilde{P}),F(\widetilde{\omega}))$, one can come back to resurvey the performance of information granularity in Definition 7. In fact, the axiomatic definition in Definition 7 is still not the best characterization of information granularity of a fuzzy granular structure. In Definition 7, one needs to find a suitable mapping function f such that $F(\widetilde{P})\widetilde{\preceq}_3 F(\widetilde{Q})$. Nevertheless, if this partial order relation cannot be found between $F(\widetilde{P})$ and $F(\widetilde{Q})$, we will not compare their information granularities. From the viewpoint of the fuzzy granular structure distance, we can overcome this limitation. In other words, for two given fuzzy granular structures, if one cannot distinguish fineness/roughness relationship in-between them, we can first use the finest fuzzy granular structure as a reference and, then, observe the fuzzy granular structure distance between every fuzzy granular structure and the finest one. The longer the fuzzy granular structure distance between a fuzzy granular structure and the finest one, the bigger the information granularity of this fuzzy granular structure. This mechanism can be closely explained by Fig. 1.

In Fig. 1, $F(\widetilde{P}), F(\widetilde{Q})$, and $F(\widetilde{R})$ are three fuzzy granular structures, and $F(\widetilde{\omega})$ are the finest fuzzy granular structures, where the partial order relations $\widetilde{\preceq}_1$, $\widetilde{\preceq}_2$, and $\widetilde{\preceq}_3$ are all not found between $F(\widetilde{P})$ and $F(\widetilde{Q})$. That is to say, each of the axiomatic definition of fuzzy rough granularity, that of generalized fuzzy rough granularity, and that of fuzzy information granularity cannot deal with this situation, whereas, if we take the finest fuzzy granular structure $F(\widetilde{\omega})$ as a reference, then the fuzzy granular structure distance can work. In particular, it is of interest that when $F(\widetilde{P})\widetilde{\preceq}_1 F(\widetilde{R})$, $D(F(\widetilde{R}),F(\widetilde{\omega})) = D(F(\widetilde{R}),F(\widetilde{P})) + D(F(\widetilde{P}),F(\widetilde{\omega}))$.

Based on the point of view, we develop a more generalized and comprehensible axiomatic definition of information granularity of a fuzzy granular structure in GrC.

Definition 8: Let $F=(U,\mathbf{R})$ be a fuzzy granular structure base, if $\forall\widetilde{P}\in\mathbf{R}$, there exists a real number $g(\widetilde{P})$ satisfying the below properties:

1) $g(\widetilde{P})\geq 0$; (Nonnegativity)
2) if $D(F(\widetilde{P}),F(\widetilde{\omega})) = D(F(\widetilde{Q}),K(\widetilde{\omega})), \forall\widetilde{P},\widetilde{Q}\in\mathbf{R}$, then $g(\widetilde{P}) = g(\widetilde{Q})$; (Invariability)
3) if $D(F(\widetilde{P}),F(\widetilde{\omega})) < D(F(\widetilde{Q}),F(\omega)), \forall\widetilde{P},\widetilde{Q}\in\mathbf{R}$, then $g(\widetilde{P}) < g(\widetilde{Q})$, (Granulation monotonicity)

then $\widetilde{g}$ is called a GFIG.

In the following, we analyze several properties of the GFIG above.

Theorem 8: Let g is a GFIG on a fuzzy granular structure base $F=(U,\mathbf{R})$, $\widetilde{P},\widetilde{Q}\in\mathbf{R}$. One has the following properties:

1) $g(\widetilde{P}) = g(\wr\wr\widetilde{P})$;
2) $g(\widetilde{P}\bigcap\widetilde{Q})\leq g(\widetilde{P})$, $g(\widetilde{P}\bigcap\widetilde{Q})\leq g(\widetilde{Q})$;
3) $g(\widetilde{P})\leq g(\widetilde{P}\bigcup\widetilde{Q})$, $g(\widetilde{Q})\leq g(\widetilde{P}\bigcup\widetilde{Q})$.

Proof: They are straightforward.

Example 6 (Continued from Example 2): To distinguish the coarseness/fineness degree between those two fuzzy granular structures, we, respectively, calculate two fuzzy granular structure distances to the finest fuzzy granular structure $F(\widetilde{\omega})$ as follows:

$$D(F(\widetilde{P}),F(\widetilde{\omega})) = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_P(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$$

$$= \frac{1+0.3+0.6+0.4+0.1}{16} = \frac{3}{20}$$

and

$$D(F(\widetilde{Q}),F(\widetilde{\omega})) = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$$

$$= \frac{1+0.6+0.7+0.3+0.7+0.8+0.7+0.4}{16} = \frac{13}{40}$$

.

Obviously, one has that $D(F(\widetilde{P}),F(\widetilde{\omega})) < D(F(\widetilde{Q}),F(\widetilde{\omega}))$. Hence, the coarseness/fineness between these two fuzzy granular structures can be distinguished, and $F(\widetilde{Q})$ is much coarser than $F(\widetilde{P})$. Therefore, the axiomatic definition of GFIG is much better than that of fuzzy information granularity in Definition 7.

In next study, we address whether each of GK in Definition 2 and E_r in Definition 3 satisfies the proposed axiomatic definition of GFIG or not.

Theorem 9: GK in Definition 2 is a GFIG under Definition 8.

Proof:

1) Obviously, it is nonnegative.
2) Let $F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$ be two fuzzy granular structures, where $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))$, $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1), G_{\widetilde{Q}}(x_2), \ldots, G_{\widetilde{Q}}(x_n))$. We assume that $D(F(\widetilde{P}), F(\widetilde{\omega})) = D(F(\widetilde{Q}), F(\widetilde{\omega}))$; then, one has that

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|} = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|},$$

that is,

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)|-0}{|U|} = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)|-0}{|U|}$$

hence, $\sum_{i=1}^{|U|}|G_{\widetilde{P}}(x_i)| = \sum_{i=1}^{|U|}|S_{\widetilde{Q}}(x_i)|$; therefore, $\widetilde{P} \approx \widetilde{Q}$ (see the definition of "$\approx$" in Section III-B). Then, from the definition of $\widetilde{\preceq}_3$, we can know that there exists a sequence $F^{'}(\widetilde{Q})$ of $F(\widetilde{Q})$, where $F^{'}(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1^{'}), G_{\widetilde{Q}}(x_2^{'}), \ldots, G_{\widetilde{Q}}(x_n^{'}))$, such that $|G_{\widetilde{P}}(x_i)| = |G_{\widetilde{Q}}(x_i^{'})|,\ i \le n$. Therefore

$$GK(\widetilde{P}) = \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{P}}(x_i)|}{n} = \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{Q}}(x_i^{'})|}{n} = \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{Q}}(x_i)|}{n} = GK(\widetilde{Q})$$

3) If $D(F(\widetilde{P}), F(\widetilde{\omega})) < D(F(\widetilde{Q}), F(\widetilde{\omega}))$, i.e., $\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|} < \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|}$, that is, $\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)|-0}{|U|} < \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)|-0}{|U|}$; hence, $\sum_{i=1}^{|U|}|G_{\widetilde{P}}(x_i)| < \sum_{i=1}^{|U|}|G_{\widetilde{Q}}(x_i)|$; therefore, $\widetilde{P}\widetilde{\preceq}_3\widetilde{Q}$, then there exists a sequence $F^{'}(\widetilde{Q})$ of $F(\widetilde{Q})$, where $F^{'}(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1^{'}), G_{\widetilde{Q}}(x_2^{'}), \ldots, G_{\widetilde{Q}}(x_n^{'}))$, such that $|G_{\widetilde{P}}(x_i)| \le |G\widetilde{Q}(x_i^{'})|,\ i \le n$, and there exists $x_0 \in U$ such that $|G_{\widetilde{P}}(x_0)| < |G_{\widetilde{Q}}(x_0^{'})|$. Hence,

$$\begin{aligned}GK(\widetilde{P}) &= \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{P}}(x_i)|}{n}\\ &= \frac{1}{n}\left(\sum_{i=1,x_i\neq x_0}^{n}\frac{|G_{\widetilde{P}}(x_i)|}{n} + \frac{|G_{\widetilde{P}}(x_0)|}{n}\right)\\ &< \frac{1}{n}\left(\sum_{i=1,x_i\neq x_0}^{n}\frac{|G_{\widetilde{Q}}(x_i^{'})|}{n} + \frac{|G_{\widetilde{Q}}(x_0^{'})|}{n}\right)\\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{|G_{\widetilde{Q}}(x_i)|}{n} = GK(\widetilde{Q}),\end{aligned}$$

that is, $GK(\widetilde{P}) < GK(\widetilde{Q})$.

Summarizing the above, GK in Definition 2 is a GFIG under Definition 8. The proof is complete. ■

Theorem 10: E_r in Definition 3 is a GFIG under Definition 8.

Proof:

1) Obviously, it is nonnegative.
2) Let $F(\widetilde{P}), F(\widetilde{Q}) \in \mathbf{F}(U)$ be two fuzzy granular structures, where $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))$, $F(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1), G_{\widetilde{Q}}(x_2), \ldots, G_{\widetilde{Q}}(x_n))$. We assume that $D(F(\widetilde{P}), F(\widetilde{\omega})) = D(F(\widetilde{Q}), F(\widetilde{\omega}))$; then,

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|} = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|},$$

that is,

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)|-0}{|U|} = \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)|-0}{|U|};$$

hence, $\sum_{i=1}^{|U|}|G_{\widetilde{P}}(x_i)| = \sum_{i=1}^{|U|}|G_{\widetilde{Q}}(x_i)|$; therefore, $\widetilde{P} \approx \widetilde{Q}$ (see the definition of "$\approx''$ in Section III-B), then there exists a sequence $F^{'}(\widetilde{Q})$ of $F(\widetilde{Q})$, where $F^{'}(\widetilde{Q}) = (G_{\widetilde{Q}}(x_1^{'}), G_{\widetilde{Q}}(x_2^{'}), \ldots, G_{\widetilde{Q}}(x_n^{'}))$, such that $|G_{\widetilde{P}}(x_i)| = |G_{\widetilde{Q}}(x_i^{'})|,\ i \le n$. Therefore

$$\begin{aligned}E_r(\widetilde{P}) &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|G_{\widetilde{P}}(x_i)|}\\ &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|G_{\widetilde{Q}}(x_i^{'})|}\\ &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{Q}}(x_i)|} = E_r(\widetilde{Q}).\end{aligned}$$

3) If $D(F(\widetilde{P}), F(\widetilde{\omega})) < D(F(\widetilde{Q}), F(\widetilde{\omega}))$, then

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|} < \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)\Delta G_{\widetilde{\omega}}(x_i)|}{|U|},$$

that is,

$$\frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{P}}(x_i)|-0}{|U|} < \frac{1}{|U|}\sum_{i=1}^{|U|}\frac{|G_{\widetilde{Q}}(x_i)|-0}{|U|}$$

hence, $\sum_{i=1}^{|U|}|G_{\widetilde{P}}(x_i)| < \sum_{i=1}^{|U|}|G_{\widetilde{Q}}(x_i)|$; therefore, there exists a sequence $F^{'}(\widetilde{Q})$ of $F(\widetilde{Q})$, where $F^{'}(\widetilde{Q}) =$

$(G_{\widetilde{Q}}(x_1'), G_{\widetilde{Q}}(x_2'), \ldots, G_{\widetilde{Q}}(x_n'))$, such that $|G_{\widetilde{P}}(x_i)| \leq |G_{\widetilde{Q}}(x_i')|$, $i \leq n$, and there exists $x_0 \in U$ such that $|G_{\widetilde{P}}(x_0)| < |G_{\widetilde{Q}}(x_0')|$. Hence

$$\begin{aligned} E_r(\widetilde{P}) &= -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{|G_{\widetilde{P}}(x_i)|} = \sum_{i=1}^{n} \frac{1}{n} \log_2 |G_{\widetilde{P}}(x_i)| \\ &= \frac{1}{n} \sum_{i=1, x_i \neq x_0}^{n} \log_2 |G_{\widetilde{P}}(x_i)| + \frac{1}{n} \log_2 |G_{\widetilde{P}}(x_0)| \\ &< \frac{1}{n} \sum_{i=1, x_i \neq x_0}^{n} \log_2 |G_{\widetilde{Q}}(x_i)| + \frac{1}{n} \log_2 |G_{\widetilde{Q}}(x_0)| \\ &= -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{|G_{\widetilde{P}}(x_i)|} = E_r(\widetilde{Q}) \end{aligned}$$

i.e., $E_r(\widetilde{P}) < E_r(\widetilde{Q})$.

From the above, we conclude that E_r in Definition 3 is a GFIG under Definition 8. The proof is complete. ■

Remark: Based on these analysis and discussions above, one can draw such a conclusion: The GFIG defined by the fuzzy granular structure distance can well distinguish the coarseness/fineness degree between any two fuzzy granular structures from the same universe, which can completely solve the problem of how to measure the information granularity of a fuzzy granular structure in fuzzy-set-based GrC. These results will be very significant for studying uncertainty in GrC.

VI. Connecting Fuzzy Information Granularity and Fuzzy Information Granularity by Fuzzy Granular Structure Distance

The concept of entropy is originally from Physics, which is often used to assess out-of-order of a system. The bigger the entropy value of a system is, the higher the out-of-order of this system is. In information theory, the notion of entropy was introduced by Shannon to measure uncertainty of a system's structure [44]. The entropy in information theory is named information entropy. It is well known that the information entropy can well measure the information content of an information system. The extended version of the entropy to measure information content of a fuzzy granular structure is called a fuzzy information entropy. There has been two forms of fuzzy information entropy in the existing literature works [3], [5], [6], [41]. In this section, with the viewpoint of the fuzzy granular structure distance, we want to reveal the connection between fuzzy information granularity and fuzzy information entropy.

To measure the uncertainty of a fuzzy granular structure, the Shannon' entropy was extended by Hu *et al.* [3], and this variant was also used to characterize the uncertainty of a fuzzy rough set and that of a fuzzy probability rough sets. The variant could overcome the limitation of Shannon's entropy only working in classical sets.

Definition 9 (see[3]): Let $F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2), \ldots, G_{\widetilde{R}}(x_n))$; then, fuzzy information entropy of $F(\widetilde{R})$ is defined as

$$H(\widetilde{R}) = -\frac{1}{n} \sum_{i=1}^{n} \log_2 \frac{|G_{\widetilde{R}}(x_i)|}{n}. \tag{12}$$

When $F(\widetilde{R})$ is a Pawlak granular structure, the fuzzy information entropy will have the same form as Shannon's entropy. In other words, the uncertainty of a Pawlak granular structure also can be calculated by this definition with a uniform configuration.

Through generalizing the Liang's mutual information entropy, Qian *et al.* [41] gave another measure for fuzzy granular structures. This measure can also be used to measure the uncertainty of a given fuzzy granular structure. The following definition gives the form of the fuzzy information entropy.

Definition 10: Let $F(\widetilde{R}) = (G_{\widetilde{R}}(x_1), G_{\widetilde{R}}(x_2), \ldots, G_{\widetilde{R}}(x_n))$; then, fuzzy information entropy of $F(\widetilde{R})$ is defined as

$$E(\widetilde{R}) = \sum_{i=1}^{n} \frac{1}{n} \left(1 - \frac{|G_{\widetilde{R}}(x_i)|}{n}\right). \tag{13}$$

If a Pawlak granular structure be considered, the fuzzy information entropy can also be degenerated to the form of Liang's mutual entropy. The definition of the fuzzy information entropy and that of Liang's mutual entropy are constructed with a uniform configuration.

Now, we come back to consider another interesting property of the fuzzy granular structure distance. For any fuzzy granular structure $F(\widetilde{P})$, we observe the relationship among $F(\widetilde{P})$, the finest one and the coarsest one with a view of the fuzzy granular structure distance from the same universe. The following theorem is an interesting phenomenon.

Theorem 11: Let $\widetilde{\mathbf{F}}(U)$ be a family of all fuzzy granular structures from a given universe U, and $F(\widetilde{P})$ a fuzzy granular structure in $\widetilde{\mathbf{F}}(U)$. Then, $D(F(\widetilde{P}), F(\widetilde{\delta})) + D(F(\widetilde{P}), F(\widetilde{\omega})) = 1$.

Proof: Let $F(\widetilde{P}) = (G_{\widetilde{P}}(x_1), G_{\widetilde{P}}(x_2), \ldots, G_{\widetilde{P}}(x_n))$, where $G_{\widetilde{P}}(x_i) = p_{i1}/x_1 + p_{i2}/x_2 + \cdots + p_{in}/x_n$, $F(\widetilde{\delta}) = (G_{\widetilde{\delta}}(x_1), G_{\widetilde{\delta}}(x_2), \ldots, G_{\widetilde{\delta}}(x_n))$, where $G_{\widetilde{\delta}}(x_i) = 1/x_1 + 1/x_2 + \cdots + 1/x_n$, and $F(\widetilde{\omega}) = (G_{\widetilde{\omega}}(x_1), G_{\widetilde{\omega}}(x_2), \ldots, G_{\widetilde{\omega}}(x_n))$, where $G_{\widetilde{\omega}}(x_i) = 0/x_1 + 0/x_2 + \cdots + 0/x_n$.

From the definition of fuzzy granular structure distance, one has

$$D(F(\widetilde{P}), F(\widetilde{\delta})) = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}$$

and

$$D(F(\widetilde{P}), F(\widetilde{\omega})) = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{i=1}^{n} (1 - p_{ij}).$$

Hence, we have

$$D(F(\widetilde{P}), F(\widetilde{\delta})) + D(F(\widetilde{P}), F(\widetilde{\omega})) = 1.$$

Summarizing the above, this completes the proof. ■

Theorem 11 indicates that the fuzzy granular structure distance of a fuzzy granular structure to the coarsest one and that of this granular structure to the finest one are strictly complementary. Thinking about the characteristic of the GFIG with the fuzzy granular structure distance in the previous section, we then explore some connections between fuzzy information entropy and the fuzzy granular structure distance.

First, we address the relationship between each of two fuzzy information entropies of a fuzzy granular structure and the fuzzy granular structure distance between it and the finest granular structure. From the definitions of fuzzy information entropy and the fuzzy granular structure distance, we obtain the following two theorems.

Theorem 12: Let $\widetilde{\mathbf{F}}(U)$ be a family of all fuzzy granular structures from a given universe U, and $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$. If $D(F(\widetilde{P}), F(\widetilde{\omega})) \geq D(F(\widetilde{Q}), F(\widetilde{\omega}))$, then $H(\widetilde{P}) \leq H(\widetilde{Q})$.

Proof: If $D(F(\widetilde{P}), F(\widetilde{\omega})) \geq D(F(\widetilde{Q}), F(\widetilde{\omega}))$, from the proof of Theorem 10, one easily has $|G_{\widetilde{P}}(x_i)| \geq |G_{\widetilde{Q}}(x_i^{'})|,\ i \leq n$, and according to Definition 9, then $H(\widetilde{P}) \leq H(\widetilde{Q})$ holds. ■

Theorem 13: Let $\widetilde{\mathbf{F}}(U)$ be a family of all fuzzy granular structures from a given universe U, and $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$. If $D(F(\widetilde{P}), F(\widetilde{\omega})) \geq D(F(\widetilde{Q}), F(\widetilde{\omega}))$, then $E(\widetilde{P}) \leq E(\widetilde{Q})$.

Proof: Similar to the proof of Theorem 12, it can be proved. ■

Motivated by the strictly complementary in Theorem 11, it is very interesting to observe the relationship between each of two fuzzy information entropies of a fuzzy granular structure and the fuzzy granular structure distance between it and the coarsest granular structure. From the definitions of fuzzy information entropy and the fuzzy granular structure distance, we draw the following two conclusions.

Theorem 14: Let $\widetilde{\mathbf{F}}(U)$ be a family of all fuzzy granular structures from a given universe U, and $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$. If $D(F(\widetilde{P}), F(\widetilde{\delta})) \leq D(F(\widetilde{Q}), F(\widetilde{\delta}))$, then $H(\widetilde{P}) \leq H(\widetilde{Q})$.

Proof: The theorem follows directly from Theorems 11 and 12 and Definition 9.

Theorem 15: Let $\widetilde{\mathbf{F}}(U)$ be a family of all fuzzy granular structures from a given universe U, and $F(\widetilde{P}), F(\widetilde{Q}) \in \widetilde{\mathbf{F}}(U)$. If $D(F(\widetilde{P}), F(\widetilde{\delta})) \leq D(F(\widetilde{Q}), F(\widetilde{\delta}))$, then $E(\widetilde{P}) \leq E(\widetilde{Q})$.

Proof: The theorem can be directly proved from Theorem 11 and 12 and Definition 10.

From the above five theorems, we could say that in a sense, there may exist a complement relationship between fuzzy information granularity and fuzzy information entropy. That is to say, they could have the same capability on measuring the uncertainty of a fuzzy granular structure on the same universe. The fuzzy granular structure distance plays a key role for building this bridge between fuzzy information granularity and fuzzy information entropy in GrC.

VII. Applicable Analysis

The fuzzy granular structure distance and the GFIG have some potential applications. For example, in rough set theory, the GFIG can help us to effectively choose suitable fuzzy granular structures for approximating a target concept or a target decision with much higher approximation accuracy. For another example, the fuzzy granular structure distance can be used to construct a heuristic function in the process of feature selection, and be also used to perform association analysis between two variables. In order to the compactness of the article, we do not make a detailed discussion here.

In what follows, we only analyze the application effectiveness of the proposed fuzzy granular structure distance and the GFIG in the GrC area. To address this issue, we conduct two kinds of numerical experiments with 12 real-world datasets coming from UCI Repository of machine learning databases, which are shown as Table I. In this table, Glass Identification, Ecoli, Pima Indians Diabetes, Seeds, Planning Relax, and Wine are six numeric datasets; and Breast Cancer, Lenses, Balloons, Space Shuttle Autolanding Domain, Hayes-Roth, and Soybean are six categorical datasets.

Before testing how the fuzzy granular structure distance behaves in real-world applications, we need to generate first fuzzy granular structures from these six datasets. For categorical data, the partition induced by a set of features is regarded as one special fuzzy granular structures. For the six numeric datasets, we normalize the numerical feature a into the interval $[0,1]$ with $a^{'} = \frac{a - a_{\min}}{a_{\max} - a_{\min}}$.

The value of the fuzzy similarity degree r_{ij} in (1) between objects x_i and x_j with respect to feature a is calculated as

$$r_{ij} = \begin{cases} 1 - 4 \times |x_i - x_j|, |x_i - x_j| \leq 0.25; \\ 0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

As $r_{ij} = r_{ji}$ and $r_{ii} = 1$, $0 \leq r_{ij} \leq 1$, the matrix M in (1) is a fuzzy similarity relation. The fuzzy similarity relation determines a fuzzy binary granular structure. Given a dataset with m features, one can generate 2^m fuzzy binary granular structures (see final column in Table I), which are used to test the effectiveness of the proposed fuzzy granular structure distance and GFIG.

In first experiment, we compare the fuzzy granular structure distance with information granularity for differentiating fuzzy granular structures coming from the same dataset. We compute pairs of fuzzy granular structures differentiated by the fuzzy granular structure distance and information granularity, respectively. With loss of generality, we select GK in (8) as one representative of the information granularity family in this experiment. These experimental results on these 12 datasets are shown in Fig. 2. The index Identifiable ratio is computed by the formula $Identifiable\ ratio = \frac{pairs\ of\ fuzzy\ granular\ structures\ distinguished\ from\ each\ other}{all\ pairs\ of\ fuzzy\ granular\ structures}$.

It is easy to see from Fig. 2 that on each dataset, the Identifiable ratio of the fuzzy granular structure distance is equal to or greater than that of the information granularity GK. This shows that compared with information granularity, the proposed fuzzy granular structure distance has much better performance for characterizing differences among fuzzy granular structures. In fact, as long as two given fuzzy granular structures are not the same as each other, they can be distinguished by the fuzzy granular structure distance.

In second experiment, we compare the GFIG with three existing versions for characterizing coarseness/fineness degrees of fuzzy granular structures coming from the same dataset. We compute pairs of fuzzy granular structures whose

TABLE I
TWELVE DATASETS IN THE EXPERIMENTAL ANALYSIS

Datasets	Objects	Features	Granular structures	Pairs of granular structures
Glass Identification	214	9	511	130 305
Ecoli	336	7	127	8001
Pima Indians Diabetes	768	8	255	32 385
Seeds	210	7	127	8001
Planning Relax	182	12	4095	8 382 465
Wine	178	13	8191	33 542 145
Breast Cancer	699	9	511	130 305
Lenses	24	4	15	105
Balloons	20	4	15	105
Space Shuttle Autolanding Domain	15	6	63	1953
Hayes-Roth	132	4	15	105
Soybean (Small)	47	35	511	130 305

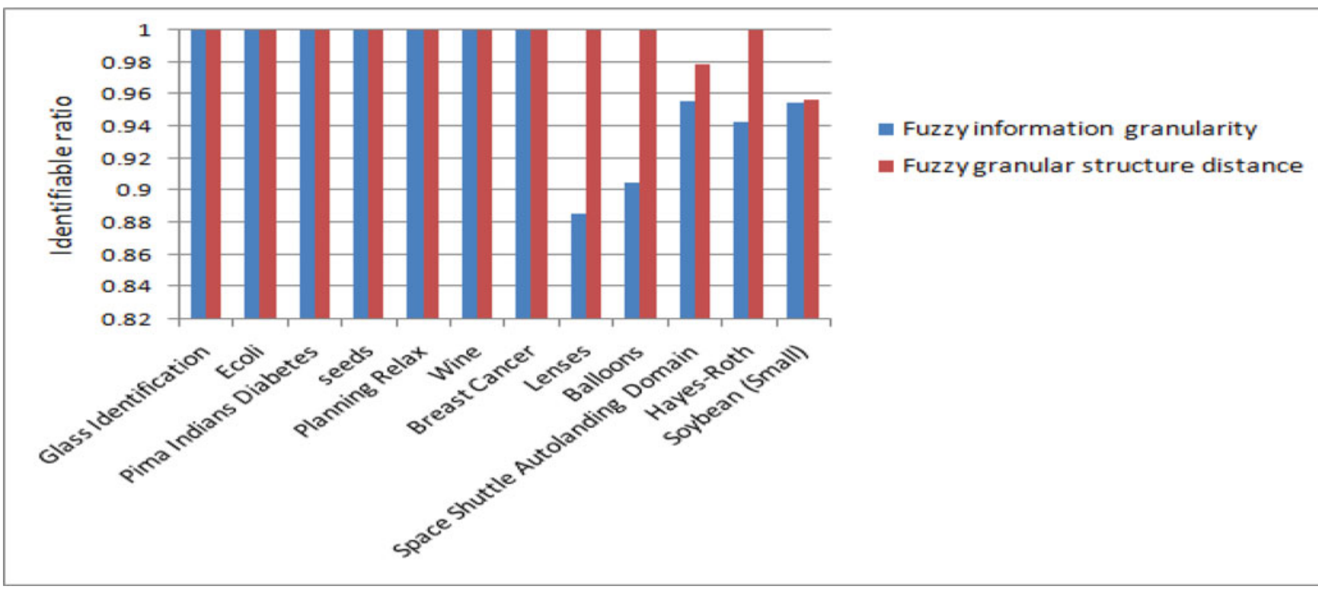

Fig. 2. Comparison on fuzzy granular structure distance and information granularity.

coarseness/fineness degrees can be characterized by the GFIG and three existing versions, respectively. Experimental results on the 12 datasets are shown in Fig. 3.

From Fig. 3, one can see that the Identifiable ratio of the GFIG is consistently and significantly much better than each of existing axiomatic approaches to fuzzy information granularity on the 12 datasets. This implies that compared with FRG, GFRG, and FIG, GFIG possesses much stronger ability for comparing coarseness/fineness relationships among fuzzy granular structures. These four axiomatic approaches to fuzzy information granularity can be ranked as follows:

$$\text{FRG} \rightarrow \text{GFRG} \rightarrow \text{FIG} \rightarrow \text{GFIG}.$$

GFIG is the best one in these four axiomatic approaches to fuzzy information granularity. It is worth pointing out that even though two given fuzzy granular structures are the same as each other in the sense of GFIG, the granular structure distance between them may be still not zero.

VIII. CONCLUSION

In fuzzy GrC proposed by Zadeh, a fuzzy granular structure means a mathematical structure of the collection of fuzzy information granules granulated from a dataset. The concept of fuzzy information granularity is employed to measure the uncertainty of a fuzzy granular structure. However, in order to profoundly study uncertainty in fuzzy GrC, we have analyzed two limitations of the fuzzy information granularity. The first limitation is that the fuzzy information granularity cannot well distinguish the difference between any two fuzzy granular structures. This arises from the fact that when the information granularity of one fuzzy granular structure is equal to that of the other, this does not mean that these two fuzzy granular structures are equivalent to each other. The second limitation is that the existing axiomatic definitions of fuzzy information granularity are still not able to well measure the coarseness/fineness relationships among some fuzzy granular structures. To address these issues, we have proposed a so-called fuzzy granular structure distance in this study,

Fig. 3. Pairs of fuzzy granular structures distinguished by four different axiomatic approaches to fuzzy information granularity respectively and their Identifiable ratios on 12 datasets.

which can well discriminate the difference between any two fuzzy granular structures. Unlike fuzzy information granularity, as long as two fuzzy granular structures is different, it must make a fine distinction between them. This is because that the fuzzy granular structure distance takes both the distribution of all fuzzy information granules and the difference between the two fuzzy information granules induced by each object into account. To solve the second limitation, based on the proposed fuzzy granular structure distance, we have developed a generalized axiomatic approach to fuzzy information granularity, under which the coarseness/fineness of any two fuzzy granular structures can be distinguished. In this approach, the partial order relation among fuzzy granular structures is established by the fuzzy granular structure distance between each fuzzy granular structure and the finest one. It is very interesting that through taking the fuzzy granular structure distances of a fuzzy granular structure to the finest one, and the coarsest one into account, we have also built a bridge between fuzzy information granularity and fuzzy information entropy. These results will be very significant for studying uncertainty in fuzzy GrC.

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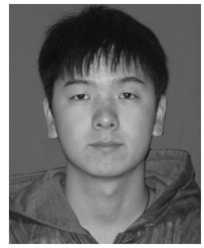

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