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A fuzzy multigranulation decision-theoretic approach to multi-source fuzzy information systems

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ABSTRACT

Decision-theoretic rough set theory (*DTRS*) is becoming one of the important research directions for studying set approximations using Bayesian decision procedure and probability theory in rough set community. In this paper, a novel model, fuzzy multigranulation decision-theoretic rough set model (*FM-DTRS*), is proposed in terms of inclusion measure of fuzzy rough sets in the viewpoint of fuzzy multigranulation. Gaussian kernel is used to compute the similarity between objects, which induces a fuzzy equivalence relation, and then we make use of T_p-norm operator with the property of Hadamart product to aggregate the multiple induced fuzzy equivalence relations. We employ the aggregated relation to fuzzily partition the universe and then obtain multiple fuzzy granulations from the multi-source information system. Moreover, some of its properties are addressed. A comparative study between the proposed fuzzy multigranulation decision-theoretic rough set model and Qian's multigranulation decision-theoretic rough set model is made. An example is employed to illustrate the effectiveness of the proposed method which may provide an effective approach for multi-source data analysis in real applications.

1. Introduction

Rough set theory, originated by Pawlak [27,28], is a mathematical tool to deal with uncertainty in a wide variety of applications [2–4,9,18,19,29,30,42,43,50]. In the past 10 years, several extensions of Pawlak rough set model have been proposed in terms of various requirements, such as the decision-theoretic rough set model [43], the variable precision rough set (*VPRS*) model [56], the rough set model based on tolerance relation [12,36], the Bayesian rough set model [39], the fuzzy rough set model [6] and the probabilistic rough sets [44]. The probabilistic rough sets, as an important research direction in rough set community, have been paid close attentions [10,11,13,14,44–47,49]. Specially, Yao [44] presented a new rule induction method based on the decision-theoretic rough set allowing for error tolerance through setting the thresholds: α and β, which is constructed by positive region, boundary region and negative region, respectively. Since then, the decision-theoretic rough sets have attracted more and more concerns. Azam and Yao [1] proposed a threshold configuration mechanism for reducing the overall uncertainty of probabilistic regions in the probabilistic rough sets. Jia et al. [10] developed an optimization representation of decision-theoretic rough set model, and gave a heuristic approach and a particle swarm optimization approach for implementing an attribute reduction with a minimum cost. Liu et al. [13,15] combined the logistic regression with the decision-theoretic rough set to form a new classification approach and investigated the three-way decision procedure with incomplete information combining the incomplete information table and loss function table together. Yu et al. [49] applied decision-theoretic rough set model to automatically determining the number of clusters with much smaller time cost. Li et al. [20] developed a sequential strategy in a decision process, which based on a formal description of granular computing.

In the view of granular computing (proposed by Zadeh [51]), in the existing rough set models, a concept described by a set is always characterized via the so-called upper and lower approximations under a single granulation, i.e., the concept is depicted by known knowledge induced from a single binary relation on the universe. Conveniently, this kind of rough set models is called single granulation rough sets. Based on a user's different requirements, Qian et al. [32] developed the multigranulation rough set which provides a new perspective for decision making analysis based on the rough set theory. Since the multigranulation rough set was proposed, the theoretical framework has been largely enriched, and many extended

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multigranulation rough set models and relative applications have also been proposed and studied [17,21–24,32,33,38,40,41]. For example, Qian et al. [33] have first proposed a new multigranulation rough set model through combining MGRS and the decision-theoretic rough sets together, called a multigranulation decision-theoretic rough set model. Sang et al. [31] proposed a new decision-theoretic rough set model based on the local rough set and the dynamic granulation principle, called a decision-theoretic rough set under dynamic granulation (DG-DTRS) which satisfies the monotonicity of the positive region of a target concept (or decision).

However, the decision and most of knowledge in the real life applications are often fuzzy and one often encounters a kind of special information system in which data come from different sources, such an information system is called a multi-source information system. Therefore it is necessary to introduce the fuzzy rough methodology into the classical *DTRS* for wider applications. The researchers [4,6,16,52–55] dealt with the real-value data sets by applying a fuzzy rough technique to solving problem. For example, Chen et al. [4] and Zhao et al. [53] used fuzzy rough sets to propose novel methods for attribute reduction and rule induction. Liang et al. [16] has proposed the triangular fuzzy decision-theoretic rough set by considering the losses being expressed by triangular fuzzy numbers. However, they still cannot be used to analyze data in the context of fuzzy multigranulation, which limits its further applications in many problems under the framework of the fuzzy environment. This motivates us to develop a new approximate strategy based on decision-theoretic rough sets to analyze data from the multi-source fuzzy information system.

Kernel methods have been proven to be an important methodology which is widely discussed in pattern recognition and machine learning domains. It maps data into a higher dimensional feature space in order to simplify classification tasks and make them linear [37,42]. In the rough set field, Hu et al. [7,8] found a high level of similarity between kernel methods and rough sets and made use of kernel to extract fuzzy relations for rough sets based data analysis. As an example, in this paper, Gaussian kernel is used to generate a fuzzy binary relation which satisfies reflexive, symmetric and transitive. However, the existing study is based on data coming from only a single source and little attention was paid to deal with data which come from different sources. To address this issue, in this study, we will introduce Gaussian kernel to extract a fuzzy equivalence relation between objects from a multi-source information system. It can be proven that the similarity matrix induced by Gaussian kernel is both reflexive and positive semi-definitive. Then we employ the T_P-norm operator: $T_P(a, b) = a \cdot b$ to aggregate multiple induced fuzzy relations to get an aggregation matrix which is called Hadamard product matrix [35]. It can induce a new fuzzy T_{cos}-equivalence relation because it is still both reflexive and positive semi-definitive, and then it will be used to partition a universe into a family fuzzy information granules forming a fuzzy granular structure for one of the sources from the multi-source information system. By the same way, one can obtain multiple fuzzy granular structures which constitute the fundamentals of the novel model in the paper.

From the above, based on different fuzzy granular structures generating from a multi-source fuzzy information system, the aim of this paper is to present a new approach to approximate the decision class with a certain level of tolerance for errors through inclusion measure between two fuzzy granules. This approach called fuzzy multigranulation decision-theoretic rough sets (*FM-DTRS*) combines the multigranulation decision-theoretic idea with fuzzy set theory. Some of its properties are addressed. A comparative study between the proposed and Qian's multigranulation decision-theoretic rough set model is made.

The rest of this paper is organized as follows. Some basic concepts of classical rough sets, variable precision fuzzy rough sets, and multigranulation rough sets are briefly reviewed in Section 2. In Section 3, we first investigate two fuzzy multigranulation decision-theoretic rough set forms that include the optimistic fuzzy multigranulation decision-theoretic rough sets, and the pessimistic fuzzy multigranulation decision-theoretic rough sets. Then, we analyze the loss function and the entire decision risk in the context of fuzzy multigranulation. When the thresholds have a special constraint, the multigranulation decision-theoretic rough sets will produce one of various variables of multigranulation rough sets. In Section 4, an example is used to illustrate our method. Finally, Section 5 concludes this paper by bringing some remarks and discussions.

2. Preliminaries

In this section, we introduce some basic notions and redescribe some related rough set models by inclusion degree, which are Pawlak rough sets, variable precision fuzzy rough sets, and multigranulation decision-theoretic rough sets, repestively [7,27,34,52,56]. Throughout this paper, let U be a finite non-empty set called the universe of discourse. The class of all fuzzy sets in U will be denoted as $F(U)$. For a set A, $|A|$ denotes the cardinality of the set A.

Definition 1. Assumed $\widetilde{R}$ is a fuzzy equivalence relation induced by a numerical attribute or fuzzy attribute. For any $x, y, z \in U$, it satisfies:

(1) reflexivity: $\widetilde{R}(x, y) = 1$;
(2) Symmetry: $\widetilde{R}(x, y) = \widetilde{R}(y, x)$; and
(3) Transitivity: $\widetilde{R}(x, z) \geq min(\widetilde{R}(x, y), \widetilde{R}(y, z))$.

The relation can be written as a matrix as

$$M(\widetilde{R}) = (\widetilde{r}_{ij})_{n\times n} = \begin{bmatrix} \widetilde{r}_{11} & \widetilde{r}_{12} & \cdots & \widetilde{r}_{1n} \\ \widetilde{r}_{21} & \widetilde{r}_{22} & \cdots & \widetilde{r}_{2n} \\ \vdots & & \ddots & \\ \widetilde{r}_{n1} & \widetilde{r}_{n2} & \cdots & \widetilde{r}_{nn} \end{bmatrix}$$

where $\widetilde{r}_{ij}$ is the similarity degree between x_i and x_j.

If condition (3) is replaced by $T(\widetilde{R}(x, y), \widetilde{R}(y, z)) \leq \widetilde{R}(x, z)$ called (T-transitivity), then $\widetilde{R}$ is said to be a fuzzy T-equivalence relation Kerre and Ovchinnikov where T is some triangular norm .

Definition 2. The fuzzy equivalence class $S_{\widetilde{R}}(x_i)$ of x_i induced by a fuzzy equivalence relation $\widetilde{R}$ is defined as

$$S_{\widetilde{R}}(x_i) = \frac{\widetilde{r}_{i1}}{x_1} + \frac{\widetilde{r}_{i2}}{x_2} + \cdots + \frac{\widetilde{r}_{in}}{x_n},$$

where '$+$' means the union operation. Obviously, $S_{\widetilde{R}}(x_i)$ is a fuzzy information granule containing x_i. $\widetilde{r}_{ij}$ is the degree of x_i equivalent to x_j. Obviously, a crisp equivalence class $[x]_R$ is a special fuzzy granule with $\widetilde{r}(x, y) = 1, y \in [x]_R$.

As we know, a fuzzy equivalence relation generates a family of fuzzy information granules from the universe, which composes a fuzzy equivalence granular structure, written by $K(\widetilde{R}) = \{S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n)\}$. Particularly, if $\widetilde{r}_{ii} = 1$ and $\widetilde{r}_{ij} = 0, j \neq i, i, j < n$, then $S_{\widetilde{R}}(x_i) = 1, i \leq n$, and $\widetilde{R}$ is called a fuzzy identity relation, and we write it as $\widetilde{R} = \varphi$; if $\widetilde{r}_{ij} = 1, i, j < n$, then $|S_{\widetilde{R}}(x_i)| = |U|, i \leq n$ and $\widetilde{R}$ is called a fuzzy universal relation, which is written as $\widetilde{R} = \delta$.

Definition 3. [7] Let A and B be two fuzzy granules in the universe U, the inclusion measure $I(A, B)$) is defined as

$$I(A, B) = \frac{|A \wedge B|}{|A|},$$

where '$\wedge$' means the operation 'min' and $|A| = \sum_{x\in U} \mu_A(x)$.

We denote $A \subset_{\varepsilon} B$, meaning $I(A, B) \geq \varepsilon$.

Pawlak rough set is based on the two fundamentals concepts: an equivalence relation R and a family of equivalence classes which is a partition of a finite non-empty universe U. If $U = \{x_1, x_2, \ldots, x_n\}$ is characterized with a collection of attribute, each attribute generates an indiscernible relation R on U. Then $< U, R >$ is called an approximation space. The family of the equivalence classes $[x]_R$ are called elemental information granules in the approximation space. They form a family of concepts to approximate arbitrary subset of objects. Based on the inclusion measure, the lower, upper approximations and boundary region of $X \subseteq U$ can also be defined as

$$\underline{R}(X) = \{x \in U \mid I([x]_R, X) = 1\},$$

$$\overline{R}(X) = \{x \in U \mid I([x]_R, X) > 0\},$$

$$BN(X) = \{x \in U \mid 0 < I([x]_R, X) < 1\}.$$

Since the equivalence relation appears to be too strict, which limits the applicability of Pawlak rough set model, Ziarko [56] first introduced an error-tolerance level with set inclusion [52] to propose the concept of variable precision rough set model (*VPRS*) which is defined as

$$\underline{R}^{\alpha}(X) = \{x \in U \mid I([x]_R, X) \geq \alpha\},$$

$$\overline{R}^{\beta}(X) = \{x \in U \mid I([x]_R, X) > \beta\}.$$

Furthermore, if the fuzziness is taken into consideration, then it is important to extend *VPRS* into *FVPRS*. That is, the variable precision lower and upper approximations of a crisp set X by a family of fuzzy information granules are defined as

$$\underline{\widetilde{R}}^{\alpha}(X) = \{x \in U \mid I(S_{\widetilde{R}}(x), X) \geq \alpha\},$$

$$\overline{\widetilde{R}}^{\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}}(x), X) > \beta\},$$

where $1 \geq \alpha \geq 0.5$, $0.5 > \beta \geq 0$. And the variable precision boundary region is

$$BN^{\alpha,\beta}(X) = \overline{\widetilde{R}}^{\beta}(X) - \underline{\widetilde{R}}^{\alpha}(X) = \{x \in U \mid \beta < I(S_{\widetilde{R}}(x), X) \leq \alpha\}.$$

In the classical rough set theory, the approximate operators partition the universe into three disjoint regions $POS(A)$, $NEG(A)$ and $BND(A)$. In this section, we briefly review some basic concepts in the decision-theoretic rough set. For convenience, in the Bayesian decision procedure, a finite set of states can be written as $\Omega = \{\omega, \neg\omega\}$, and a finite set of three possible actions can be denoted by $A = \{a_1, a_2, a_3\}$, where a_1, a_2 and a_3 represent the three actions, deciding *accept*, deciding *decline* and deciding *discard*. Let $P(\omega|X)$ and $P(\neg\omega|X)$ be the condition probability of an object x being in state ω and $\neg\omega$, respectively, given that the object is described by X. Let $\lambda(a_j|\omega)$ denote the loss, or cost, for taking action a_j when the state is ω, the expected loss associated with taking action a_i, $(i = 1, 2, 3)$ is given by

$$R(a_i|X) = \lambda(a_i|\omega) \cdot P(\omega|X) + \lambda(a_i|\neg\omega) \cdot P(\neg\omega|X).$$

The Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) decide $POS(\omega)$ if $R(a_1|X) \leq R(a_2|X)$ and $R(a_1|X) \leq R(a_3|X)$;
(B) decide $NEG(\omega)$ if $R(a_2|X) \leq R(a_1|X)$ and $R(a_2|X) \leq R(a_3|X)$;
(N) decide $BND(\omega)$ if $R(a_3|X) \leq R(a_1|X)$ and $R(a_3|X) \leq R(a_2|X)$.

Based on Yao's study, the minimum-risk decision rules (P),(N),(D) can be written as:

(P)decide $POS(\omega)$ if $P(\omega|X) \geq \alpha$ and $P(\omega|X) \geq \gamma$;
(B)decide $NEG(\omega)$ if $P(\omega|X) \leq \beta$ and $P(\omega|X) \leq \gamma$;
(N)decide $BND(\omega)$ if $P(\omega|X) \leq \alpha$ and $P(\omega|X) \geq \beta$;

where $\alpha = \frac{\lambda_{12}-\lambda_{32}}{(\lambda_{31}-\lambda_{32})-(\lambda_{11}-\lambda_{12})}$, $\beta = \frac{\lambda_{32}-\lambda_{22}}{(\lambda_{21}-\lambda_{22})-(\lambda_{31}-\lambda_{32})}$, $\gamma = \frac{\lambda_{12}-\lambda_{22}}{(\lambda_{21}-\lambda_{22})-(\lambda_{11}-\lambda_{12})}$. And $\lambda_{11} = \lambda(a_1|\omega), \lambda_{21} = \lambda(a_2|\omega), \lambda_{31} = \lambda(a_3|\omega), \lambda_{12} = \lambda(a_1|\neg\omega), \lambda_{22} = \lambda(a_2|\neg\omega), \lambda_{32} = \lambda(a_3|\neg\omega)$.

Through using the conditional probability $P(\omega|X)$, the Bayesian decision procedure can decide how to assign x into these three disjoint regions [44].

If one takes multiple binary equivalence relations into problem solving, the multigranulation rough set will be proposed. According to two different approximation strategies, Qian et al. [32,33] developed two different multigranulation rough sets (*MGRS*) including the optimistic and pessimistic ones.

Definition 4. Let $IS = (U, AT, f)$ be a complete information system, $A_1, A_2, \ldots, A_m \subseteq AT$, and $X \subseteq U$. The optimistic lower and upper approximations of X with respect to $A_1, A_2, \ldots, A_m$ are denoted by $\underline{\sum_{i=1}^{m} A_i}^{O} X$ and $\overline{\sum_{i=1}^{m} A_i}^{O} X$, respectively, where

$$\underline{\sum_{i=1}^{m} A_i}^{O,\alpha=1}(X) = \{x \in U \mid I([x]_{A_1}, X) = 1 \vee I([x]_{A_2}, X) = 1 \vee \cdots \vee I([x]_{A_m}, X) = 1\},$$

$$\overline{\sum_{i=1}^{m} A_i}^{O,\beta=0}(X) = \{x \in U \mid I([x]_{A_1}, X) > 0 \wedge I([x]_{A_2}, X) > 0 \wedge \cdots \wedge I([x]_{A_m}, X) > 0\}.$$

Then $(\underline{\sum_{i=1}^{m} A_i}^{O}(X), \overline{\sum_{i=1}^{m} A_i}^{O}(X))$ is called the classical optimistic MGRS [32].

In addition, the definition of the classical pessimistic MGRS [33] is defined as follows:

$$\underline{\sum_{i=1}^{m} A_i}^{P,\alpha=1}(X) = \{x \in U \mid I([x]_{A_1}, X) = 1 \wedge I([x]_{A_2}, X) = 1 \wedge \cdots \wedge I([x]_{A_m}, X) = 1\},$$

$$\overline{\sum_{i=1}^{m} A_i}^{P,\beta=0}(X) = \{x \in U \mid I([x]_{A_1}, X) > 0 \vee I([x]_{A_2}, X) > 0 \vee \cdots \vee I([x]_{A_m}, X) > 0\}.$$

3. Fuzzy multigranulation decision-theoretic rough sets

Let us consider a scenario when one obtains information regarding a set of objects from different sources. Each source is regarded as a classical information system which has some attributes with fuzzy attribute values. Thus, such an information system is called a multi-source fuzzy information system.

Definition 5. *A multi-source information system* is $MS = \{IS_i \mid IS_i = (U, AT_i, \{(V_a)_{a \in AT_i}\}, f_i)\}$, where,

(1) U is a finite non-empty set of objects, called the universe;
(2) AT_i is a non-empty finite set of attributes of each subsystem;
(3) $\{V_a\}$ is the value of the attribute $a \in AT_i$; and
(4) f_i: $U \times AT_i \to \{(V_a)_{a \in AT_i}\}$ such that for all $x \in U$ *and* $a \in AT_i$, $f(x, a) \in V_a$.

Particular, if the attribute value is fuzzy, we call $MS = \{IS_i \mid IS_i = (U, AT_i, \{(V_a)_{a \in AT_i}\}, f_i)\}$ a multi-source fuzzy information system. In this paper, we suppose MS is composed of m single-source information systems. Similar to the granulation method for the single-source information system, one gets m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, where $K(\widetilde{R}_i) = \{S_{\widetilde{R}_i}(x_1), S_{\widetilde{R}_i}(x_2), \ldots, S_{\widetilde{R}_i}(x_n)\}$ and $S_{\widetilde{R}_i}(x_j)\{j = 1, 2, \ldots, n\}$ are fuzzy granules. As a result, one obtains a fuzzy multigranulation approximation space, denoted as $(U, K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m))$.

In this section, how to approximate X through m fuzzy condition granular structures and how to extract fuzzy decision rules in a given multi-source fuzzy system are two important problems in the process of multi-source data analysis. It has been proven that a fuzzy binary relation over U with Gaussian kernel satisfies reflexive, symmetric and T_{cos}-transitive, called a fuzzy T-equivalence relation over U [25]. In the following, Gaussian kernel is introduced to acquire a fuzzy T-equivalence relation which will partition a universe into a family of fuzzy information granules.

Suppose U is a finite set of objects, $x_i \in U$ is described by a vector $< x_{i1}, x_{i2}, \ldots, x_{im} > \in R_n$. Thus, U is viewed as a subset of R_n.

The similarity between two objects is computed by Gaussian kernel $K(x_i, x_j) = exp(-\frac{\|x_i - x_j\|^2}{2\delta^2})$, where $\|x_i - x_j\|$ is the Euclidean distance between x_i and x_j and $K(x_i, x_j)$ satisfies

(1) $K(x_i, x_j) \in [0, 1]$;
(2) $K(x_i, x_j) = K(x_j, x_i)$; and
(3) $K(x_i, x_i) = 1$.

Lemma 1 ([25]). *A fuzzy binary relation induced by Gaussian kernel is T_{cos}-equivalence relation, where $T_{cos}(a, b) = max\{a \cdot b - \sqrt{1-a^2} \cdot \sqrt{1-b^2}, 0\}$.*

Lemma 2 ([26]). *A matrix induced by Gaussian kernel is positive semi-definite.*

Lemma 3 ([35]). *Let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in F_{n\times n}$ be two positive semi-definite matrices, then Hadamard product $A \circ B$ is positive semi-definite, where $A \circ B = (a_{ij} * b_{ij})_{n\times n}$.*

Lemma 3 is quite important and instrumental to achieve the main results of this paper. It is possible for us to get a new relation satisfying reflexivity, symmetry and T-transitivity by the property of Hadamard product.

Theorem 1. *Let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in F_{n\times n}$ be two reflexive (i.e., $a_{ii} = 1, i = 1, 2, \ldots, n$) and positive semi-definite matric, then Hadamard product $A \circ B$ is reflexive and positive semi-definite, where $A \circ B = (a_{ij} * b_{ij})_{n\times n}$.*

Proof. Suppose $C = A \circ B = (c_{ij})_{n\times n}$, based on the definition of Hadamard product $A \circ B$, we know that $c_{ii} = a_{ii} * b_{ii} = 1$. Hence, C is reflexive. And the matrix C is positive semi-definite matric in terms of Lemma 3. Therefore, it holds. □

Theorem 2. *A reflexive and positive semi-definite matrix is equivalent to a fuzzy T_{cos}- equivalence relation.*

Proof. It can be proved easily by Lemma 1, Lemma 2 and Lemma 3. □

From the above discussions, we use $\widetilde{R}^G(x_i, x_j) = exp(-\frac{\|x_i - x_j\|^2}{2\delta^2})$ to compute the similarity of objects x_i and x_j with respect to an attribute of a single source information system and then obtains a corresponding similarity matrix which is both reflexive and positive semi-definitive. In terms of Theorem 2, we use T_p-norm operator: $T_p(a, b) = a \cdot b$ to get a Hadamard product matrix which is still both reflexive and positive semi-definitive. Therefore, an aggregated fuzzy equivalence relation will be induced by the Hadamard product matrix and then granulate the universe U into a family of fuzzy equivalence information granules forming a fuzzy granular structure of a single information system. By using the same method, one obtains a family of fuzzy granular structures with respect to different sources from the multi-source information system.

3.1. Optimistic fuzzy multigranulation decision-theoretic rough sets

Using the idea of the optimistic multigranulation rough sets: a target concept's multigranulation lower approximation only needs at least one granular structure to satisfy with the inclusion condition between an equivalence class and the approximation target, the equation $\overline{R}(X) = 1 - NEG(X)$, and the breaking-criteria [44], we give a crisp concept's approximations through m fuzzy granular structures in terms of fuzzy T-equivalence relations computed with Gaussian kernel.

Definition 6. Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$, the optimistic lower and upper approximations of X, denoted by $\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X)$ and $\overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X)$, respectively, are defined as

$$\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) \geq \alpha \vee I(S_{\widetilde{R}_2}(x), X) \geq \alpha \vee \cdots \vee I(S_{\widetilde{R}_m}(x), X) \geq \alpha\},$$

$$\overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X) = U - \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) \leq \beta \vee I(S_{\widetilde{R}_2}(x), X) \leq \beta \vee \cdots \vee I(S_{\widetilde{R}_m}(x), X) \leq \beta\},$$

where $S_{\widetilde{R}_i}(x)$ is the fuzzy equivalence class of x induced by the fuzzy T-equivalence relations $\widetilde{R}_i$, $I([x]_{\widetilde{R}_i}, X)$ is the fuzzy inclusion degree between $[x]_{\widetilde{R}_i}$ and X, and α, β are two probability constraints with $0.5 \leq \alpha \leq 1$ and $0 \leq \beta < 0.5$.

Then, we call $(\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X), \overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X))$, the optimistic fuzzy multigranulation decision-theoretic rough sets. By the lower and upper approximations, the fuzzy optimistic multigranulation decision-theoretic boundary region of X is defined as

$$BN^{o,\alpha,\beta}_{\sum_{i=1}^{m} \widetilde{R}_i}(X) = \overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X) - \underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X).$$

Theorem 3. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$, the optimistic upper approximation of X of the fuzzy multigranulation decision-theoretic rough set is also represented as*

$$\overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) > \beta \wedge I(S_{\widetilde{R}_2}(x), X) > \beta \wedge \cdots \wedge I(S_{\widetilde{R}_m}(x), X) > \beta\}.$$

From the definition of optimistic fuzzy multigranulation decision-theoretic rough sets, one can obtain the following propositions.

Proposition 1. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X) \supseteq \underline{\widetilde{R}_i}^{\alpha}(X)$;
(2) $\overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X) \subseteq \overline{\widetilde{R}_i}^{\alpha}(X)$;

where

$$\underline{\widetilde{R}}^{\alpha}(X) = \{x \in U \mid I(S_{\widetilde{R}}(x), X) \geq \alpha\},$$

$$\overline{\widetilde{R}}^{\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}}(x), X) < \beta\}.$$

Proposition 2. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X) = \bigcup_{i=1}^{m} \underline{\widetilde{R}_i}^{\alpha}(X)$;
(2) $\overline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\beta}(X) = \bigcap_{i=1}^{m} \overline{\widetilde{R}_i}^{\beta}(X)$.

Proposition 3. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and two crisp decision classes $X \subseteq Y \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(X) \supseteq \underline{\sum_{i=1}^{m} \widetilde{R}_i}^{O,\alpha}(Y)$;

(2) $\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{O,\beta}(X) \subseteq \overline{\sum_{i=1}^{m}\widetilde{R}_i}^{O,\beta}(Y)$.

The optimistic fuzzy multigranulation decision-theoretic rough sets are the rational extension of some models. Let us derive the other model from its definitions.

Case 1. If X is a crisp subset and $\widetilde{R}_i, (i = 1, 2, \ldots, m)$ are m crisp equivalence relations on U, the optimistic *FM-DTRS* is degenerated to *MDTRS*. Here, it deserves special noting here that the equation (19) in [34] which is the upper optimistic multigranulation approximation of X, should be corrected as follows:

$$\overline{\sum_{i=1}^{m} R_i}^{O,\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) > \beta \wedge I(S_{\widetilde{R}_2}(x), X) > \beta \wedge \cdots \wedge I(S_{\widetilde{R}_m}(x), X) > \beta\}.$$

Case 2. If X is a crisp subset and $\widetilde{R}_i, (i = 1)$ is a crisp equivalence relation on U, the optimistic *FM-DTRS* is degenerated to *DTRS*.

3.1.1. Optimistic fuzzy multigranulation loss function

Based on the Bayesian decision procedure and Yao's decision-theoretic study, let $R_k(a_i|X)$ be kth expected loss under kth granular structure, then the optimistic fuzzy multigranulation decision's expected loss associated with taking action a_1, a_2, a_3 is given by

$$R_{\sum_{k=1}^{m}\widetilde{R}_k^{(O,\alpha,\beta)}} = \sum_{k=1}^{m} R_k(a_i|X), (i = 1, 2, 3).$$

3.1.2. Optimistic fuzzy multigranulation decision rules

Similar to the classical decision-theoretic rough sets, when the thresholds $1 \geq \alpha > \beta \geq 0$, we can get the following decision rules:

(OFMP1) if $\exists i \in \{1, 2, \ldots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \geq \alpha$, decide *POS*($X$);
(OFMB1) if $\forall i \in \{1, 2, \ldots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \leq \beta$, decide *NEG*($X$);
(OFMN1) otherwise, decide *BND*(X).

When the thresholds $1 \geq \alpha = \gamma = \beta \geq 0$, we can get the following decision rules:

(OFMP1) if $\exists i \in \{1, 2, \ldots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \geq \alpha$, decide *POS*($X$);
(OFMB1) if $\forall i \in \{1, 2, \ldots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \leq \alpha$, decide *NEG*($X$);
(OFMN1) otherwise, decide *BND*(X).

3.2. Pessimistic fuzzy multigranulation decision-theoretic rough sets

Definition 7. Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$, the pessimistic lower and upper approximations of X, denoted by $\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X)$ and $\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X)$, respectively, are defined as

$$\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) \geq \alpha \wedge I(S_{\widetilde{R}_2}(x), X) \geq \alpha \wedge \cdots \wedge I(S_{\widetilde{R}_m}(x), X) \geq \alpha\},$$

$$\overline{\sum_{i=1}^{m} R_i}^{P,\beta}(X) = U - \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) \leq \beta \wedge I(S_{\widetilde{R}_2}(x), X) \leq \beta \wedge \cdots \wedge I(S_{\widetilde{R}_m}(x), X) \leq \beta\}.$$

By the lower and upper approximations, the fuzzy pessimistic multigranulation decision-theoretic boundary region of X is defined as

$$BN^{P,\alpha,\beta}_{\sum_{i=1}^{m}\widetilde{R}_i}(X) = \overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X) - \underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X).$$

Then, we call $(\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X), \overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X))$ pessimistic fuzzy multigranulation decision-theoretic rough sets, and α, β are two probability constraints with $0.5 \leq \alpha \leq 1$ and $0 \leq \beta < 0.5$.

Theorem 4. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$, the pessimistic upper approximation of X of the fuzzy multigranulation decision-theoretic rough set is also represented as*

$$\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) > \beta \vee I(S_{\widetilde{R}_2}(x), X) > \beta \vee \cdots \vee I(S_{\widetilde{R}_m}(x), X) > \beta\}.$$

From the definition of fuzzy pessimistic multigranulation decision-theoretic rough sets, one can obtain the following propositions.

Proposition 4. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X) \subseteq \underline{\widetilde{R}_i}^{\alpha}(X)$;
(2) $\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X) \subseteq \overline{\widetilde{R}_i}^{\beta}(X)$.

Proposition 5. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and a crisp decision class $X \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X) = \bigcap_{i=1}^{m}\underline{\widetilde{R}_i}^{\alpha}(X)$;
(2) $\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X) = \bigcup_{i=1}^{m}\overline{\widetilde{R}_i}^{\beta}(X)$.

Proposition 6. *Given m fuzzy granular structures: $K(\widetilde{R}_1), K(\widetilde{R}_2), \ldots, K(\widetilde{R}_m)$, and two crisp decision classes $X \subseteq Y \subseteq U$. Then the following properties hold.*

(1) $\underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(X) \supseteq \underline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\alpha}(Y)$;
(2) $\overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(X) \subseteq \overline{\sum_{i=1}^{m}\widetilde{R}_i}^{P,\beta}(Y)$.

The pessimistic fuzzy multigranulation decision-theoretic rough sets are the rational extension of some models. Let us derive the other model from its definitions.

Case 1. If X is a crisp subset and $\widetilde{R}_i, (i = 1, 2, \ldots, m)$ are m crisp equivalence relation on U, the pessimistic *FM-DTRS* is degenerated to *MDTRS*. Here, it deserves special noting here that the equation (22) in [34] which is upper pessimistic multigranulation approximation of X, should be corrected as follows:

$$\overline{\sum_{i=1}^{m} R_i}^{P,\beta}(X) = \{x \in U \mid I(S_{\widetilde{R}_1}(x), X) > \beta \vee I(S_{\widetilde{R}_2}(x), X) > \beta \vee \cdots \vee I(S_{\widetilde{R}_m}(x), X) > \beta\}.$$

Case 2. If X is a crisp subset and $\widetilde{R}_i, (i = 1)$ is a crisp equivalence relation on U, the pessimistic *FM-DTRS* is degenerated to *DTRS*.

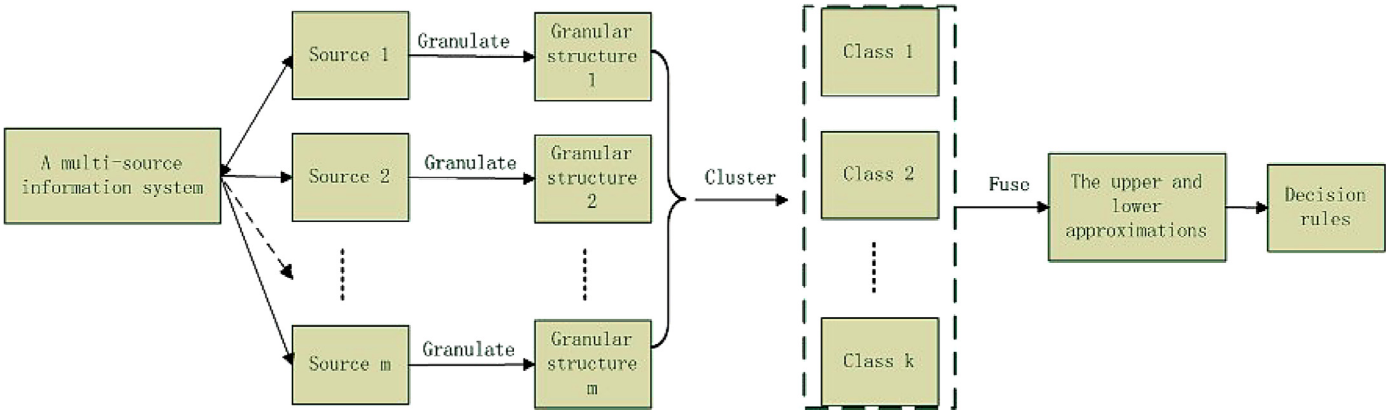

Fig. 1. Structure of *FM-DTRS*.

Table 1
A multi-source fuzzy information system of an evaluation problem of a credit card applicant.

	EC_1			EC_2			EC_3		*Decision*	
	$c1$	$c4$	$c7$	$c2$	$c5$	$c8$	$c3$	$c6$	$c9$	D
x_1	0.8	0.1	0.2	0.1	0.1	0.5	0.1	0.2	0.3	Accept
x_2	0.3	0.5	0.2	0.3	0.3	0.7	0.4	0.2	0.1	Accept
x_3	0.2	0.1	0.6	0.6	0.3	0.2	0.2	0.6	0.2	Decline
x_4	0.6	0.3	0.5	0.2	0.2	0.2	0.2	0.5	0.3	Accept
x_5	0.4	0.4	0.3	0.3	0.4	0.3	0.3	0.2	0.4	Decline
x_6	0.2	0.3	0.5	0.2	0.3	0.6	0.2	0.4	0.2	Decline
x_7	0.3	0.3	0.6	0.3	0.4	0.2	0.3	0.3	0.2	Accept
x_8	0.3	0.4	0.3	0.3	0.2	0.2	0.3	0.4	0.5	Decline
x_9	0.3	0.2	0.4	0.3	0.4	0.4	0.4	0.4	0.2	Decline

3.2.1. Pessimistic fuzzy multigranulation loss function

Based on the Bayesian decision procedure and Yao's decision-theoretic study, let $R_k(a_i|X)$ be kth expected loss under kth granular structure, then the pessimistic fuzzy multigranulation decision's expected loss associated with taking action a_1, a_2, a_3 is given by

$$R_{\sum_{k=1}^{m}\widetilde{R}_k^{(P,\alpha,\beta)}} = \Pi_{k=1}^{m} R_k(a_i|X), \; (i = 1, 2, 3).$$

3.2.2. Pessimistic fuzzy multigranulation decision rules

Similar to the classical decision-theoretic rough sets, when the thresholds $1 \ge \alpha > \beta \ge 0$, we can get the following decision rules:

(PFMP1) if $\forall i \in \{1, 2, \ \ldots, \ m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \ge \alpha$, decide $POS(X)$;
(PFMB1) if $\exists i \in \{1, 2, \ \ldots, \ m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \le \beta$, decide $NEG(X)$;
(PFMN1) otherwise, decide $BND(X)$.

When the thresholds $1 \ge \alpha = \gamma = \beta \ge 0$, we can get the following decision rules:

(PFMP1) if $\forall i \in \{1, 2, \ldots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \ge \alpha$, decide $POS(X)$;
(PFMB1) if $\exists i \in \{1, 2, \cdots, m\}$ such that $I(S_{\widetilde{R}_i}(x), X) \le \alpha$, decide $NEG(X)$;
(PFMN1) otherwise, decide $BND(X)$.

Based on the above discussions, we can gain a structure of *FM-DTRS*, which is shown as Fig. 1.

4. Example

In this section, we employ an example to illustrate our proposed method [5,48].

Example 1. Let us consider an evaluation problem of a credit card applicants depicted by a fuzzy multi-source decision information system. Suppose that $U = \{x_1, x_2, \ \ldots, \ x_9\}$ is a set of nine applicants. Every applicant in each sub-information (source) system, denoted by EC_1, EC_2, and EC_3, is described by three fuzzy conditional attributes. They are $c1 =$ *best education*, $c4 =$ *high salary*, $c7 =$ *older age*, $c2 =$ *better education*, $c5 =$ *middle salary*, $c8 =$ *middle age*, $c3 =$ *good education*, $c6 =$ *low salary*, and $c9 =$ *young age*, respectively. The membership degrees of every applicant are given in Table 1. A decision partition is $D_1 = \{x_1, x_2, x_4, x_7\}$ and $D_2 = \{x_3, x_5, x_6, x_8, x_9\}$.

In what follows, we will describe the process of computing in detail.

(1) We make use of Gaussian kernel with an assumption $\delta = 0.3$ to compute the similarity degree to induce a fuzzy T-equivalence relation between x_i $(i = 1, 2, \ldots, 9)$ and x_j $(j = 1, 2, \ \ldots, \ 9)$ by each attribute of every source. Then for a source EC_1 of the multi-source information

system, one obtains three similarity matrix induced from conditional attributes: $c1$, $c4$, and $c7$, respectively, which are as follows:

$$(R^G_{c1}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.2494 & 0.1353 & 0.8007 & 0.4111 & 0.1353 & 0.2494 & 0.2494 & 0.2494 \\ 0.2494 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 0.9460 & 1.0000 & 1.0000 & 1.0000 \\ 0.1353 & 0.9460 & 1.0000 & 0.4111 & 0.8007 & 1.0000 & 0.9460 & 0.9460 & 0.9460 \\ 0.8007 & 0.6065 & 0.4111 & 1.0000 & 0.8007 & 0.4111 & 0.6065 & 0.6065 & 0.6065 \\ 0.4111 & 0.9460 & 0.8007 & 0.8007 & 1.0000 & 0.8007 & 0.9460 & 0.9460 & 0.9460 \\ 0.1353 & 0.9460 & 1.0000 & 0.4111 & 0.8007 & 1.0000 & 0.9460 & 0.9460 & 0.9460 \\ 0.2494 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 0.9460 & 1.0000 & 1.0000 & 1.0000 \\ 0.2494 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 0.9460 & 1.0000 & 1.0000 & 1.0000 \\ 0.2494 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 0.9460 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix}$$

$$(R^G_{c4}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.4111 & 1.0000 & 0.8007 & 0.6065 & 0.8007 & 0.8007 & 0.6065 & 0.9460 \\ 0.4111 & 1.0000 & 0.4111 & 0.8007 & 0.9460 & 0.8007 & 0.8007 & 0.9460 & 0.6065 \\ 1.0000 & 0.4111 & 1.0000 & 0.8007 & 0.6065 & 0.8007 & 0.8007 & 0.6065 & 0.9460 \\ 0.8007 & 0.8007 & 0.8007 & 1.0000 & 0.9460 & 1.0000 & 1.0000 & 0.9460 & 0.9460 \\ 0.6065 & 0.9460 & 0.6065 & 0.9460 & 1.0000 & 0.9460 & 0.9460 & 1.0000 & 0.8007 \\ 0.8007 & 0.8007 & 0.8007 & 1.0000 & 0.9460 & 1.0000 & 1.0000 & 0.9460 & 0.9460 \\ 0.8007 & 0.8007 & 0.8007 & 1.0000 & 0.9460 & 1.0000 & 1.0000 & 0.9460 & 0.9460 \\ 0.6065 & 0.9460 & 0.6065 & 0.9460 & 1.0000 & 0.9460 & 0.9460 & 1.0000 & 0.8007 \\ 0.9460 & 0.6065 & 0.9460 & 0.9460 & 0.8007 & 0.9460 & 0.9460 & 0.8007 & 1.0000 \end{bmatrix}$$

$$(R^G_{c7}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 1.0000 & 0.4111 & 0.6065 & 0.9460 & 0.6065 & 0.4111 & 0.9460 & 0.8007 \\ 1.0000 & 1.0000 & 0.4111 & 0.6065 & 0.9460 & 0.6065 & 0.4111 & 0.9460 & 0.8007 \\ 0.4111 & 0.4111 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 1.0000 & 0.6065 & 0.8007 \\ 0.6065 & 0.6065 & 0.9460 & 1.0000 & 0.8007 & 1.0000 & 0.9460 & 0.8007 & 0.9460 \\ 0.9460 & 0.9460 & 0.6065 & 0.8007 & 1.0000 & 0.8007 & 0.6065 & 1.0000 & 0.9460 \\ 0.6065 & 0.6065 & 0.9460 & 1.0000 & 0.8007 & 1.0000 & 0.9460 & 0.8007 & 0.9460 \\ 0.4111 & 0.4111 & 1.0000 & 0.9460 & 0.6065 & 0.9460 & 1.0000 & 0.6065 & 0.8007 \\ 0.9460 & 0.9460 & 0.6065 & 0.8007 & 1.0000 & 0.8007 & 0.6065 & 1.0000 & 0.9460 \\ 0.8007 & 0.8007 & 0.8007 & 0.9460 & 0.9460 & 0.9460 & 0.8007 & 0.9460 & 1.0000 \end{bmatrix}$$

(2) We use T_p-norm: $T_p(a,b) = a * b$ to aggregate the three similarity degrees and gets a similarity matrix on three attributes of EC_1, which generates a fuzzy partition called a fuzzy granular structure on U as follows:

$$(R_{EC_1}^G(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.1025 & 0.0556 & 0.3889 & 0.2359 & 0.0657 & 0.0821 & 0.1431 & 0.1889 \\ 0.1025 & 1.0000 & 0.1599 & 0.2946 & 0.8465 & 0.4594 & 0.3292 & 0.8948 & 0.4857 \\ 0.0556 & 0.1599 & 1.0000 & 0.3114 & 0.2946 & 0.7575 & 0.7575 & 0.3480 & 0.7165 \\ 0.3889 & 0.2946 & 0.3114 & 1.0000 & 0.6065 & 0.4111 & 0.5738 & 0.4594 & 0.5427 \\ 0.2359 & 0.8465 & 0.2946 & 0.6065 & 1.0000 & 0.6065 & 0.5427 & 0.9460 & 0.7165 \\ 0.0657 & 0.4594 & 0.7575 & 0.4111 & 0.6065 & 1.0000 & 0.8948 & 0.7165 & 0.8465 \\ 0.0821 & 0.3292 & 0.7575 & 0.5738 & 0.5427 & 0.8948 & 1.0000 & 0.5738 & 0.7575 \\ 0.1431 & 0.8948 & 0.3480 & 0.4594 & 0.9460 & 0.7165 & 0.5738 & 1.0000 & 0.7575 \\ 0.1889 & 0.4857 & 0.7165 & 0.5427 & 0.7165 & 0.8465 & 0.7575 & 0.7575 & 1.0000 \end{bmatrix}$$

From the granular structure $(R_{EC_1}^G(x_i,x_j))$, one can get nine fuzzy information granules on U as follows:

$$S_{R_{EC_1}^G}(x_1) = \frac{1.000}{x_1} + \frac{0.1025}{x_2} + \frac{0.0566}{x_3} + \frac{0.3889}{x_4} + \frac{0.2359}{x_5} + \frac{0.0657}{x_6} + \frac{0.0821}{x_7} + \frac{0.1431}{x_8} + \frac{0.1889}{x_9};$$

$$S_{R_{EC_1}^G}(x_2) = \frac{0.1025}{x_1} + \frac{1.0000}{x_2} + \frac{0.1599}{x_3} + \frac{0.2946}{x_4} + \frac{0.8465}{x_5} + \frac{0.4594}{x_6} + \frac{0.3292}{x_7} + \frac{0.8948}{x_8} + \frac{0.4857}{x_9};$$

$$S_{R_{EC_1}^G}(x_3) = \frac{0.0556}{x_1} + \frac{0.1599}{x_2} + \frac{1.0000}{x_3} + \frac{0.3114}{x_4} + \frac{0.2946}{x_5} + \frac{0.7575}{x_6} + \frac{0.7575}{x_7} + \frac{0.3480}{x_8} + \frac{0.7165}{x_9};$$

$$S_{R_{EC_1}^G}(x_4) = \frac{0.3889}{x_1} + \frac{0.2946}{x_2} + \frac{0.3114}{x_3} + \frac{1.0000}{x_4} + \frac{0.6065}{x_5} + \frac{0.4111}{x_6} + \frac{0.5738}{x_7} + \frac{0.4594}{x_8} + \frac{0.5427}{x_9};$$

$$S_{R_{EC_1}^G}(x_5) = \frac{0.2359}{x_1} + \frac{0.8465}{x_2} + \frac{0.2946}{x_3} + \frac{0.6065}{x_4} + \frac{1.0000}{x5} + \frac{0.6065}{x6} + \frac{0.5427}{x7} + \frac{0.9460}{x8} + \frac{0.7165}{x9};$$

$$S_{R_{EC_1}^G}(x_6) = \frac{0.0657}{x1} + \frac{0.4594}{x2} + \frac{0.7575}{x3} + \frac{0.4111}{x4} + \frac{0.6065}{x5} + \frac{1.0000}{x_6} + \frac{0.8948}{x_7} + \frac{0.7165}{x_8} + \frac{0.8465}{x_9};$$

$$S_{R_{EC_1}^G}(x_7) = \frac{0.0821}{x_1} + \frac{0.3292}{x_2} + \frac{0.7575}{x_3} + \frac{0.5738}{x_4} + \frac{0.5427}{x_5} + \frac{0.8948}{x_6} + \frac{1.0000}{x_7} + \frac{0.5738}{x_8} + \frac{0.7575}{x_9};$$

$$S_{R_{EC_1}^G}(x_8) = \frac{0.1431}{x1} + \frac{0.8948}{x2} + \frac{0.3480}{x3} + \frac{0.4594}{x4} + \frac{0.9460}{x5} + \frac{0.7165}{x_6} + \frac{0.5738}{x_7} + \frac{1.0000}{x_8} + \frac{0.7575}{x_9};$$

$$S_{R_{EC_1}^G}(x_9) = \frac{0.1889}{x_1} + \frac{0.4857}{x_2} + \frac{0.7165}{x_3} + \frac{0.5427}{x_4} + \frac{0.7165}{x_5} + \frac{0.8465}{x_6} + \frac{0.7575}{x_7} + \frac{0.7575}{x_8} + \frac{1.0000}{x_9}.$$

Based on the inclusion measure of fuzzy sets, we get that

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_1), D_1) = 0.70, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_2), D_1) = 0.38, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_3), D_1) = 0.29,$$

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_4), D_1) = 0.49, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_5), D_1) = 0.39, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_6), D_1) = 0.32,$$

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_7), D_1) = 0.36, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_8), D_1) = 0.35, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_9), D_1) = 0.33.$$

Similarly, one gets

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_1), D_2) = 0.30, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_2), D_2) = 0.62, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_3), D_2) = 0.71,$$

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_4), D_2) = 0.51, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_5), D_2) = 0.61, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_6), D_2) = 0.68,$$

$$I_{\delta=0.3}(S_{R_{EC_1}^G}(x_7), D_2) = 0.64, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_8), D_2) = 0.65, I_{\delta=0.3}(S_{R_{EC_1}^G}(x_9), D_2) = 0.67.$$

By using the same method, one obtains the other two similarity matrices induced by the attributes of EC_2 and EC_3. Accordingly, one gets two other fuzzy granular structures on U as follows, respectively.

$$(R^G_{EC_2}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.5134 & 0.1211 & 0.5427 & 0.3889 & 0.7165 & 0.2946 & 0.4594 & 0.4594 \\ 0.5134 & 1.0000 & 0.1512 & 0.2231 & 0.3889 & 0.8948 & 0.2359 & 0.2359 & 0.5738 \\ 0.1211 & 0.1512 & 1.0000 & 0.3889 & 0.5427 & 0.1690 & 0.5738 & 0.5738 & 0.4594 \\ 0.5427 & 0.2231 & 0.3889 & 1.0000 & 0.7165 & 0.3889 & 0.7575 & 0.9460 & 0.6065 \\ 0.3889 & 0.3889 & 0.5427 & 0.7165 & 1.0000 & 0.5427 & 0.9460 & 0.7575 & 0.9460 \\ 0.7165 & 0.8948 & 0.1690 & 0.3889 & 0.5427 & 1.0000 & 0.3679 & 0.3679 & 0.7165 \\ 0.2946 & 0.2359 & 0.5738 & 0.7575 & 0.9460 & 0.3679 & 1.0000 & 0.8007 & 0.8007 \\ 0.4594 & 0.2359 & 0.5738 & 0.9460 & 0.7575 & 0.3679 & 0.8007 & 1.0000 & 0.6412 \\ 0.4594 & 0.5738 & 0.4594 & 0.6065 & 0.9460 & 0.7165 & 0.8007 & 0.6412 & 1.0000 \end{bmatrix}$$

Similarly, based on the inclusion degree, we have that

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_1),D_1)=0.45$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_2),D_1)=0.47$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_3),D_1)=0.31$,

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_4),D_1)=0.45$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_5),D_1)=0.39$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_6),D_1)=0.46$,

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_7),D_1)=0.40$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_8),D_1)=0.42$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_9),D_1)=0.39$.

And

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_1),D_2)=0.55$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_2),D_2)=0.53$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_3),D_2)=0.69$,

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_4),D_2)=0.55$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_5),D_2)=0.61$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_6),D_2)=0.54$,

$I_{\delta=0.3}(S_{R^G_{EC_2}}(x_7),D_2)=0.60$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_8),D_2)=0.58$, $I_{\delta=0.3}(S_{R^G_{EC_2}}(x_9),D_2)=0.61$.

Similarly,

$$(R^G_{EC_3}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.4857 & 0.3679 & 0.5738 & 0.7575 & 0.7165 & 0.7165 & 0.5134 & 0.4594 \\ 0.4857 & 1.0000 & 0.3114 & 0.3889 & 0.5738 & 0.6065 & 0.8465 & 0.3114 & 0.7575 \\ 0.3679 & 0.3114 & 1.0000 & 0.8948 & 0.3114 & 0.8007 & 0.5738 & 0.4594 & 0.6412 \\ 0.5738 & 0.3889 & 0.8948 & 1.0000 & 0.5427 & 0.8948 & 0.7165 & 0.7165 & 0.7165 \\ 0.7575 & 0.5738 & 0.3114 & 0.5427 & 1.0000 & 0.6065 & 0.7575 & 0.7575 & 0.6065 \\ 0.7165 & 0.6065 & 0.8007 & 0.8948 & 0.6065 & 1.0000 & 0.8948 & 0.5738 & 0.8007 \\ 0.7165 & 0.8465 & 0.5738 & 0.7165 & 0.7575 & 0.8948 & 1.0000 & 0.5738 & 0.8948 \\ 0.5134 & 0.3114 & 0.4594 & 0.7165 & 0.7575 & 0.5738 & 0.5738 & 1.0000 & 0.5738 \\ 0.4594 & 0.7575 & 0.6412 & 0.7165 & 0.6065 & 0.8007 & 0.8948 & 0.5738 & 1.0000 \end{bmatrix}$$

Moreover, based on the inclusion degree, we compute that

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_1),D_1)=0.50$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_2),D_1)=0.52$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_3),D_1)=0.40$,

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_4),D_1)=0.42$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_5),D_1)=0.45$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_6),D_1)=0.31$,

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_7),D_1)=0.47$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_8),D_1)=0.39$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_9),D_1)=0.44$.

And,

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_1),D_2)=0.50$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_2),D_2)=0.48$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_3),D_2)=0.60$,

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_4),D_2)=0.58$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_5),D_2)=0.55$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_6),D_2)=0.69$,

$I_{\delta=0.3}(S_{R^G_{EC_3}}(x_7),D_2)=0.53$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_8),D_2)=0.61$, $I_{\delta=0.3}(S_{R^G_{EC_3}}(x_9),D_2)=0.56$.

Based on the above three fuzzy granular structures and the inclusion degree between fuzzy granules $S_{R^G_{EC_i}}(x_i)$ and $S_{R^G_{EC_i}}(x_j)$ and assume that $\alpha=0.6$, $\beta=0.35$ and $\delta=0.30$, one can get the lower and upper optimistic/pessimistic approximations of the decision concepts D_1 and D_2, respectively.

An optimistic fuzzy lower approximation of D_1: $\underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_1) = \{x_1\}$;

An optimistic fuzzy upper approximation of D_1: $\overline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_1) = \{x_1, x_2, x_4, x_5, x_7\}$.

A pessimistic fuzzy lower approximation of D_1: $\underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_1) = \phi$;

Apessimistic fuzzy upper approximation of D_1: $\overline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_1) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

An optimistic fuzzy lower approximation of D_2: $\underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_2) = \{x_2, x_3, x_5, x_6, x_7, x_8, x_9\}$;

An optimistic fuzzy upper approximation of D_2: $\overline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_2) = \{x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

A pessimistic fuzzy lower approximation of D_2: $\underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_2) = \{x_3\}$;

Apessimistic fuzzy upper approximation of D_2: $\overline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_2) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

(3) Decision rules

a. Optimistic decision rules:

(OFMP1) if $x \in \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_1)$ or $x \in U - \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_2)$, then decide *Accept*;

(OFMB1) if $x \in U - \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_1)$ or $x \in \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{O,\alpha}(D_2)$, then decide *Decline*;

(OFMN1) otherwise, decide *Retard*.

b. Pessimistic decision rules:

(PFMP1) if $x \in \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_1)$ or $x \in U - \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_2)$, then decide *Accept*;

(PFMB1) if $x \in U - \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_1)$ or $x \in \underline{\sum_{i=1}^{3}\widetilde{R}_i^G}^{P,\alpha}(D_2)$, then decide *Decline*;

(PFMN1) otherwise, decide *Retard*.

(4) We can use the triangular number employed in [16] to compute the expected loss, which is omitted here.

Similarly, we compute with assumptions $\delta = 0.2$ and $\delta = 0.1$, respectively. But we only display the results for the case of $\delta = 0.2$ as follows.

$$(R_{EC_1}^G(x_i, x_j)) = \begin{bmatrix} 1.0000 & 0.0059 & 0.0015 & 0.1194 & 0.0388 & 0.0022 & 0.0036 & 0.0126 & 0.0235 \\ 0.0059 & 1.0000 & 0.0162 & 0.0639 & 0.6873 & 0.1738 & 0.0821 & 0.7788 & 0.1969 \\ 0.0015 & 0.0162 & 1.0000 & 0.0724 & 0.0639 & 0.5353 & 0.5353 & 0.0930 & 0.4724 \\ 0.1194 & 0.0639 & 0.0724 & 1.0000 & 0.3247 & 0.1353 & 0.2865 & 0.1738 & 0.2528 \\ 0.0388 & 0.6873 & 0.0639 & 0.3247 & 1.0000 & 0.3247 & 0.2528 & 0.8825 & 0.4724 \\ 0.0022 & 0.1738 & 0.5353 & 0.1353 & 0.3247 & 1.0000 & 0.7788 & 0.4724 & 0.6873 \\ 0.0036 & 0.0821 & 0.5353 & 0.2865 & 0.2528 & 0.7788 & 1.0000 & 0.2865 & 0.5353 \\ 0.0126 & 0.7788 & 0.0930 & 0.1738 & 0.8825 & 0.4724 & 0.2865 & 1.0000 & 0.5353 \\ 0.0235 & 0.1969 & 0.4724 & 0.2528 & 0.4724 & 0.6873 & 0.5353 & 0.5353 & 1.0000 \end{bmatrix}$$

$$(R_{EC_2}^G(x_i, x_j)) = \begin{bmatrix} 1.0000 & 0.2231 & 0.0087 & 0.2528 & 0.1194 & 0.4724 & 0.0639 & 0.1738 & 0.1738 \\ 0.2231 & 1.0000 & 0.0143 & 0.0342 & 0.1194 & 0.7788 & 0.0388 & 0.0388 & 0.2865 \\ 0.0087 & 0.0143 & 1.0000 & 0.1194 & 0.2528 & 0.0183 & 0.2865 & 0.2865 & 0.1738 \\ 0.2528 & 0.0342 & 0.1194 & 1.0000 & 0.4724 & 0.1194 & 0.5353 & 0.8825 & 0.3247 \\ 0.1194 & 0.1194 & 0.2528 & 0.4724 & 1.0000 & 0.2528 & 0.8825 & 0.5353 & 0.8825 \\ 0.4724 & 0.7788 & 0.0183 & 0.1194 & 0.2528 & 1.0000 & 0.1054 & 0.1054 & 0.4724 \\ 0.0639 & 0.0388 & 0.2865 & 0.5353 & 0.8825 & 0.1054 & 1.0000 & 0.6065 & 0.6065 \\ 0.1738 & 0.0388 & 0.2865 & 0.8825 & 0.5353 & 0.1054 & 0.6065 & 1.0000 & 0.3679 \\ 0.1738 & 0.2865 & 0.1738 & 0.3247 & 0.8825 & 0.4724 & 0.6065 & 0.3679 & 1.0000 \end{bmatrix}$$

and

$$(R^G_{EC_3}(x_i,x_j)) = \begin{bmatrix} 1.0000 & 0.1969 & 0.1054 & 0.2865 & 0.5353 & 0.4724 & 0.4724 & 0.2231 & 0.1738 \\ 0.1969 & 1.0000 & 0.0724 & 0.1194 & 0.2865 & 0.3247 & 0.6873 & 0.0724 & 0.5353 \\ 0.1054 & 0.0724 & 1.0000 & 0.7788 & 0.0724 & 0.6065 & 0.2865 & 0.1738 & 0.3679 \\ 0.2865 & 0.1194 & 0.7788 & 1.0000 & 0.2528 & 0.7788 & 0.4724 & 0.4724 & 0.4724 \\ 0.5353 & 0.2865 & 0.0724 & 0.2528 & 1.0000 & 0.3247 & 0.5353 & 0.5353 & 0.3247 \\ 0.4724 & 0.3247 & 0.6065 & 0.7788 & 0.3247 & 1.0000 & 0.7788 & 0.2865 & 0.6065 \\ 0.4724 & 0.6873 & 0.2865 & 0.4724 & 0.5353 & 0.7788 & 1.0000 & 0.2865 & 0.7788 \\ 0.2231 & 0.0724 & 0.1738 & 0.4724 & 0.5353 & 0.2865 & 0.2865 & 1.0000 & 0.2865 \\ 0.1738 & 0.5353 & 0.3679 & 0.4724 & 0.3247 & 0.6065 & 0.7788 & 0.2865 & 1.0000 \end{bmatrix}$$

So, we have

$$I_{\delta=0.2}(S_{R^G_{EC_1}}(x_1),D_1)=0.93, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_2),D_1)=0.38, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_3),D_1)=0.63,$$
$$I_{\delta=0.2}(S_{R^G_{EC_1}}(x_4),D_1)=0.61, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_5),D_1)=0.32, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_6),D_1)=0.39,$$
$$I_{\delta=0.2}(S_{R^G_{EC_1}}(x_7),D_1)=0.37, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_8),D_1)=0.30, I_{\delta=0.2}(S_{R^G_{EC_1}}(x_9),D_1)=0.24.$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_1),D_1)=0.62, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_2),D_1)=0.51, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_3),D_1)=0.20,$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_4),D_1)=0.49, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_5),D_1)=0.35, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_6),D_1)=0.44,$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_7),D_1)=0.32, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_8),D_1)=0.40, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_9),D_1)=0.32.$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_1),D_1)=0.56, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_2),D_1)=0.61, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_3),D_1)=0.36,$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_4),D_1)=0.40, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_5),D_1)=0.42, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_6),D_1)=0.38,$$
$$I_{\delta=0.2}(S_{R^G_{EC_2}}(x_7),D_1)=0.50, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_8),D_1)=0.32, I_{\delta=0.2}(S_{R^G_{EC_2}}(x_9),D_1)=0.43.$$

Here, if we also assume that $\alpha = 0.6$ and $\beta = 0.35$, one can get the optimistic/pessimistic lower and upper approximations of the decision concept D_1.

That is, an optimistic fuzzy lower approximation of D_1: $\underline{\sum_{i=1}^{3} \widetilde{R}_i^G}^{O,\alpha}(D_1) = \{x_1, x_2, x_3, x_4\}$;

An optimistic fuzzy upper approximation of D_1: $\overline{\sum_{i=1}^{3} \widetilde{R}_i^G}^{O,\alpha}(D_1) = \{x_1, x_2, x_3, x_4, x_6, x_7\}$.

A pessimistic fuzzy lower approximation of D_1: $\underline{\sum_{i=1}^{3} \widetilde{R}_i^G}^{P,\alpha}(D_1) = \phi$;

Apessimistic fuzzy upper approximation of D_1: $\overline{\sum_{i=1}^{3} \widetilde{R}_i^G}^{P,\alpha}(D_1) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$.

In addition, the optimistic/pessimistic lower and upper approximations of D_2 can be gotten by the same process.

From the above, when α and β are constant, the lower of the optimistic approximate of the decision concept is becoming larger by decreasing δ. In other words, the approximate accurate is becoming larger which is more proximate to the original data. Therefore, one can choose the reasonable parameter for higher approximate accurate through a large of experiments which will be discussed in future.

5. Conclusion and discussion

In this paper, we have proposed a new model called the fuzzy multigranulation decision-theoretic rough set approach for the multi-source information system. It is in terms of the inclusion measure of fuzzy rough sets in the viewpoint of fuzzy multigranulation. A Gaussian kernel function has been used to compute similarity between objects and then fuzzily partition the universe into fuzzy granulation. Some of its important properties have been addressed. A comparative study between the proposed fuzzy multigranulation decision-theoretic rough set model and Qian's multigranulation decision-theoretic rough set model is made. An example has been employed to illustrate our method's effectiveness in real applications. It shows that the proposed approach will be helpful for dealing with multi-source data. Further research includes how to reduce redundant fuzzy granular structures in the process of rough data analysis under the fuzzy multigranulation environment and how to fuse multi-source fuzzy data with the proposed method.

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